TENSOR TRIANGULATED CATEGORIES WITH SUB-PICARD GRADING

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Abstract. We develop an axiomatizatic generalization of the classical, motivic, and equivariant stable homotopy categories. In this general setting we: explore graded anticommutativity properties of stable homotopy rings, develop a notion of cellularity, prove an analogue of the Künneth theorem and the universal coefficient theorem, and construct a version of the dual E-Steenrod algebra and formulate the precise in which sense it is a graded anticommutative Hopf algebroid. All of our work will culminate in the construction of a (homological) E-Adams spectral sequence, and we characterize its E_2 page in terms of Ext of comodules over the dual E-Steenrod algebra.

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1. Introduction

Arguably the most famous problem in modern homotopy theory is computing the stable homotopy groups of spheres. This is an extremely difficult problem. State of the art computations by Isaksen, Wang, and Xu ([13]) have only computed these groups up to dimension 90. The key tool used for these computations is the *Adams spectral sequence*, of which there are many different flavors. Given a flat, cellular commutative ring spectrum E, the E-Adams spectral spectral sequence is $\mathbb{Z}^2$-graded and has signature

$$E_2^{s,t}(X,Y) = \mathrm{Ext}^s_{E_*(E)}(E_*(X), E_{*+t}(Y)) \Longrightarrow [X, Y^\wedge_E]_*.$$

Here Ext is taken in the category of $\mathbb{Z}$-graded comodules over the dual E-Steenrod algebra. The spectral sequence was originally constructed in 1958 by Frank Adams in the case $E = H\mathbb{F}_p$ and $X = Y = S$ ([2]). While a great deal may be ascertained about the stable homotopy groups of spheres with just this spectral sequence, more is needed in order to compute beyond the first twenty or so stable stems.

In 1998, Voevodsky introduced concepts from homotopy theory into algebraic geometry, creating a new field called *motivic homotopy theory* ([29]), also called $\mathbb{A}^1$-homotopy theory. Rather than working with topological spaces, in motivic homotopy theory, the fundamental objects are varieties over some base scheme $\mathscr{S}$. The theory goes quite far, and one may construct a symmetric monoidal stable model category of motivic spectra over $\mathscr{S}$, whose homotopy category is the motivic stable homotopy category over $\mathscr{S}$.[1] Later in 2009, Dugger and Isaksen constructed a motivic version of the Adams spectral sequence in the motivic stable homotopy category ([9]). Given a flat, cellular motivic commutative ring spectrum E, the motivic E-Adams spectral spectral sequence is $\mathbb{Z}^3$-graded and has signature

$$E_2^{s,t,u}(X,Y) = \mathrm{Ext}^s_{E_{*,*}(E)}(E_{*,*}(X), E_{*+t+s,*+u}(Y)) \Longrightarrow [X, Y^\wedge_E]_{*,*}$$

(where here Ext is taken in the category of $\mathbb{Z}^2$-graded left comodules over the motivic dual E-Steenrod algebra). Dugger and Isaksen construct the spectral sequence only for the case $E = H\mathbb{F}_2$ and $X = S$. Furthermore, they leave out many details of the construction, leaving them for the reader to fill in. The work we present here was originally conceived with the aim of filling in these details and constructing the spectral sequence in the more general form given above. More general results were achieved upon pursing this aim, and the scope of our results have changed significantly.

[1]For a review of these constructions, we refer the reader to Section 2 of [30].

1.1. **Goals & Outline.** The two main goals of this paper are as follows:

(1) Provide an axiomatic generalization of classical, motivic, and equivariant stable homotopy categories.
(2) Provide a reference for the full and explicit details of the construction of the classical, motivic, and equivariant E-Adams spectral sequences, the characterization of their E_2 pages, and some basic facts about their convergence.

The idea for the generalization came to the author after reading the pair of papers [7] and [10], which roughly discuss graded commutativity properties of what one might call "sub-Picard graded symmetric monoidal categories".

We warn the reader that, as a result of goal (2), this document is primarily expository in nature. Furthermore, it aims to be a mostly self-contained reference, which accounts for its significant length. Indeed, a large portion of the results contained here constitute only slight generalizations of results already found elsewhere in the literature. Nevertheless, we believe the approach outlined here is valuable even beyond serving as a self-contained reference, as we do make several original innovations:

(1) We provide a general construction of the Adams spectral sequence which equally applies to the classical, motivic, and equivariant stable homotopy categories. This is quite flexible, for example, in the G-equivariant case: we can construct a version of the spectral sequence which intrisically keeps track of the $RO(G)$ grading, or, alternatively, could be constructed to be graded by the entirety of the Picard group of the equivariant stable homotopy category. In particular, we give a more general version of the motivic Adams spectral sequence than that found in the literature.
(2) We develop the notion of a "tensor-triangulated category with sub-Picard grading," which roughly is a category which is graded by some abelian group, symmetric monoidal, and triangulated, all in a compatible way. Along with a few extra categorical conditions, such categories provide a surprisingly powerful axiomatization of the (classical, motivic, equivariant) stable homotopy category, and a shockingly large amount of the theory therewithin can be carried out entirely in this framework.
(3) We provide an encompassing notion of "cellularity" in a tensor triangulated category with sub-Picard grading, which parallels the same notion in the motivic stable homotopy category.
(4) We work out some of the graded-commutativity properties of $\pi_*(E)$ for a commutative monoid object (E, μ, e) in a tensor triangulated category with sub-Picard grading. In particular, we provide a complete picture of the preliminary analysis given in [7, Remark 7.2].
(5) We suggest a definition for the correct notion of an "anticommutative A-graded ring" for a general abelian group A. In particular, we suggest a new candidate for the category in which the motivic Steenrod algebra is a Hopf algebroid/co-groupoid object.

This paper should be viewed as a natural successor to the nLab page on the Adams spectral sequence ([25]) written by Urs Schreiber. Indeed, this paper tells mostly the same story told there, albeit in a more general setting. Along the way, we fill in many of the details not contained there. Furthermore, we are of the opinion that the more general and categorical approach can serve to clarify and even "trivialize" many of the proofs and ideas involved here. It is the hope of the author that this document can serve as an equally valuable resource for those first learning classical, motivic, and equivariant stable homotopy theory.

We give an outline of the structure of the paper. In Section 2, we start by giving the required background for the paper, and we give a brief review of coherence for symmetric monoidal categories. We then develop the notion of tensor triangulated categories with sub-Picard grading, which will be defined in Definition 2.3. We will discuss Dugger's paper [7] which concern "sub-Picard graded symmetric monoidal categories", and we will apply some of the results from therewithin to our situation. Then we will fix such a category $\mathcal{SH}$ (with a few extra categorical conditions), which acts as an axiomatic model for the classical, motivic, and equivariant stable homotopy categories. In this category, we will be able to develop much of the theory of stable homotopy theory, in particular, we will be able to formulate the notion of A-graded stable homotopy groups $\pi_*(X)$ of objects X in $\mathcal{SH}$, as well as homology, and cohomology represented by objects in this category. We will show that (co)fiber sequences (i.e., distinguished triangles) in $\mathcal{SH}$ give rise to long exact sequences of homotopy groups, and that $\mathcal{SH}$ is equipped with an A-indexed family of "suspension" and "loop" autoequivalences.

After just this first section, we will actually have all the data needed to construct the Adams spectral sequence, yet we will not actually do so until the very end in Section 7. The goal of this spectral sequence will be to compute the A-graded abelian groups of stable homotopy classes of maps $[X, Y]_*$ between objects X and Y in $\mathcal{SH}$, by means of algebraic information about the E-homology of X and Y. This is useful because in practice, for a suitable homology theory E, it is often easier to compute E-homology than it is to compute general hom-groups. To achieve this goal, Sections 2–6 will be devoted to formulating suitable conditions on E, X, and Y under which enough structure may be captured on the E-homology groups $E_*(X)$ and $E_*(Y)$ that algebraic information about homomorphisms between them gives suitable information about the groups $[X, Y]_*$.

In Section 3, we will formulate the notion of *cellular* objects in $\mathcal{SH}$. Intuitively, these are the objects in $\mathcal{SH}$ which may be constructed by gluing together copies of spheres. In the case $\mathcal{SH}$ is the motivic stable homotopy category, these objects will correspond to the standard notion of cellular motivic spaces. In the case $\mathcal{SH}$ is the classical stable homotopy category, every object will turn out to be cellular, as a consequence of the fact that every space is weakly equivalent to a generalized cell complex. The class of cellular objects in $\mathcal{SH}$ will satisfy many very important properties, for example, given cellular objects X and Y in $\mathcal{SH}$, a map $f : X \to Y$ will be an isomorphism if and only if it induces an isomorphism on stable homotopy groups $\pi_*(f) : \pi_*(X) \to \pi_*(Y)$. Many of the important theorems and propositions presented in this paper will require some sort of cellularity condition.

In Section 4, we will discuss the theory of monoid objects in $\mathcal{SH}$, which correspond to ring spectra in stable homotopy theory. We will show that given a monoid object E in $\mathcal{SH}$, its stable homotopy groups $\pi_*(E)$ naturally form an A-graded ring, and furthermore, E-homology $E_*(-)$ will yield a functor from $\mathcal{SH}$ to the category of A-graded left modules over $\pi_*(E)$. Here a great deal of effort will be put into formulating the exact sense in which the rings $\pi_*(E)$ are *A-graded anticommutative* when E is a commutative monoid object in $\mathcal{SH}$. In particular, here we will develop the notion of *A-graded anticommutative rings*, and we will show that $\pi_*(E)$ is an A-graded anticommutative algebra over the A-graded anticommutative stable homotopy ring $\pi_*(S)$ (where S is the monoidal unit in $\mathcal{SH}$), in a suitable sense. We will also briefly discuss some of the consequences of these results in the classical, motivic, and equivariant stable homotopy categories.

In Section 5, we will prove analogues of important theorems for homology in $\mathcal{SH}$. First of all, we will prove that for E a commutative monoid object and objects X and Y in $\mathcal{SH}$, under suitable conditions we have a *Künneth isomorphism*

$$Z_*(E) \otimes_{\pi_*(E)} E_*(W) \to \pi_*(Z \otimes E \otimes W)$$

relating the Z-homology of E and the E-homology of W to the stable homotopy groups of $Z \otimes E \otimes W$. We will then take a bit to develop the theory of module objects over monoid objects in $\mathcal{SH}$, with which we will prove a generalization of the universal coefficient theorem, which will tell us that under suitable conditions, for a monoid object E in $\mathcal{SH}$ and an object X, the cohomology $E^*(X)$ of X is the dual of the homology $E_*(X)$ as a $\pi_*(E)$-module. These two theorems will be very important for our later work.

In Section 6, we will show that for nice enough commutative monoid objects E in $\mathcal{SH}$, that the E-self homology $E_*(E)$, along with the ring $\pi_*(E)$, forms an *A-graded anticommutative Hopf algebroid*, which we define to be a co-groupoid object in the category $\pi_*(S)$-$\mathbf{GCA}^A$ of A-graded anticommutative $\pi_*(S)$-algebras. This pair $(E_*(E), \pi_*(E))$ with its additional structure as a Hopf algebroid is called the *dual E-Steenrod algebra*, over which the A-graded E-homology group $E_*(X)$ of X is canonically an A-graded left comodule for each X in $\mathcal{SH}$. This will be the culmination of our efforts to place additional structure on the E-homology groups $E_*(X)$, and we will finish the section by constructing an isomorphism

$$[X, E \otimes Y]_* \cong \mathrm{Hom}^*_{E_*(E)}(E_*(X), E_*(E \otimes Y))$$

for a suitable commutative monoid object E and objects X and Y in $\mathcal{SH}$.

In Section 7, we will finally construct the $\mathbb{Z} \times A$-graded spectral sequence $(E_r^{s,a}(X,Y), d_r)$ called the *E-Adams spectral sequence for the computation of X and Y*, and we will show that under suitable conditions, its E_2 page may be characterized in terms of a graded isomorphism

$$E_2^{*,*}(X,Y) \cong \mathrm{Ext}^{*,*}_{E_*(E)}(E_*(X), E_*(Y)).$$

Furthermore, we will briefly discuss that the natural target group of this spectral sequence is the object $[X, Y_E^\wedge]_*$, where $Y_E^\wedge$ is the "E-nilpotent completion" of Y. Furthermore, we will briefly discuss some conditions under which the spectral sequence strongly converges to this target group. We can summarize all of the results in the following theorem:

Theorem 1.1. *Let (E, μ, e) be a commutative monoid object in $\mathcal{SH}$, and let X and Y be objects. Further suppose that:*

- *E is cellular and flat,*
- *X is cellular and $E_*(X)$ is a graded projective left $\pi_*(E)$-module, and*
- *Y is cellular.*

Then there exists an object $Y_E^\wedge$ in $\mathcal{SH}$ called the E-nilpotent completion of Y *and a $\mathbb{Z} \times A$-graded spectral sequence called the* E-Adams spectral sequence for the computation of $[X, Y]_*$ *with signature*

$$E_2^{s,a}(X,Y) = \mathrm{Ext}^{s,a+\mathbf{s}}_{E_*(E)}(E_*(X), E_*(Y)) \Longrightarrow [X, Y_E^\wedge]_*.$$

Furthermore, if the derived E_∞ term RE_∞ of the sequence vanishes, then this spectral sequence converges strongly to the indicated target group.

We will give all of the relevant definitions along the way.

Finally, in Section 8, we will suggest some further directions in which one can develop the theory we have set up.

We also include five appendices on the theory of (tensor) triangulated categories, A-graded abelian groups, rings, and modules, monoid objects in symmetric monoidal categories, homological (co)algebra and derived functors, and Hopf algebroids.

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2. Preliminaries

2.1. **Background.** To start, we give a brief review of the assumed background. The most important tool we require of the reader is a familiarity with category theory, and in particular additive, abelian, and (symmetric, closed) monoidal categories. We do not recall any definitions here (mostly so as not to make an already lengthy document any longer), for that we refer the reader to any standard treatment of category theory, for example, Emily Riehl's book [27], or

Mac Lane's book [15]. In particular, see chapters 7 and 9 of the latter book for a reference on (symmetric closed) monoidal categories.

When working in monoidal categories, we will nearly always be implicitly using Mac Lane's coherence theorem for monoidal categories, which was originally proven in Mac Lane's paper [16], along with a stronger version of the theorem for symmetric monoidal categories. These theorems are tedious to rigorously state, and we do not do so here (for that we refer the reader to [7, §2]), but their consequences are intuitive. Roughly, they say that there is a strong monoidal equivalence from any monoidal category to a strict monoidal category, where tensoring with the unit, the associators, and the unitors are all the identity. In the symmetric case, the theorem says in addition that in a symmetric monoidal category, any morphisms between two objects given by "formal composites" of products of unitors, associators, symmetries, and their inverses are equal if the domain and codomain of the composites have the same underlying permutation (after removing units). In practice, the most immediate consequence of these theorems is that when constructing maps and showing diagrams commute, we will nearly always suppress associators and unitors from the notation, instead taking them to be equalities. Similarly, we will assume that tensoring with the unit is the identity. This style of reasoning is essential to understanding nearly anything written here, and as such we will usually not point out when we are applying the coherence theorems. An example of where we use coherence is in the very first proof we give, in Proposition 2.7 below.

We also assume the reader is familiar with the theory of modules and bimodules over (non-commutative) rings, along with products, direct sums, and tensor products of them. In Appendix B, assuming this knowledge, we will develop the theory of *A-graded* versions of these notions, as well as some of their properties. These notions should be very familiar to any reader familiar with the standard notion of $\mathbb{Z}$ or $\mathbb{N}$-graded rings and modules. This appendix can — and perhaps should — be skipped by anyone knowledgeable in these matters.

Finally, ideally the reader should be familiar with triangulated categories, monoid objects in monoidal categories and their modules, and derived functors, although each of these topics are developed or at least reviewed in the main body of the paper or its appendices. With all of that out of the way, we may finally get to our the key definition which underlies our work.

2.2. **Triangulated categories with sub-Picard grading.** Our goal is now to construct a list of conditions which axiomatize "a stable homotopy category of spaces". To do so, we will build up the necessary definitions one-by-one. Along the way, we will discuss some of the ramifications of our definitions and how they relate to each other. Once we have defined everything needed, we will establish the axiomatization in Convention 2.6. The first definition we will need is that of a triangulated category.

Definition 2.1. A *triangulated category* $(\mathcal{C}, \Sigma, \mathcal{D})$ is the data of:

(1) An additive category $\mathcal{C}$.
(2) An additive auto-equivalence $\Sigma : \mathcal{C} \to \mathcal{C}$ called the *shift functor*.
(3) A collection $\mathcal{D}$ of *distinguished* triangles in $\mathcal{C}$, where a *triangle* is a sequence of arrows of the form

$$X \to Y \to Z \to \Sigma X.$$

Distinguished triangles are also sometimes called *cofiber sequences* or *fiber sequences.*

These data must satisfy the following axioms:

TR0 Given a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\cong"'] & Y \arrow[r] \arrow[d, "\cong"'] & Z \arrow[r] \arrow[d, "\cong"'] & \Sigma X \arrow[d, "\cong"'] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X' \end{tikzcd}$$

where the vertical arrows are isomorphisms, if the top row is distinguished then so is the bottom.

TR1 For any object X in $\mathcal{C}$, the diagram

$$X \xrightarrow{\mathrm{id}_X} X \to 0 \to \Sigma X$$

is a distinguished triangle.

TR2 For all $f : X \to Y$ there exists an object C_f (also sometimes denoted Y/X) called the *cofiber of* f and a distinguished triangle

$$X \xrightarrow{f} Y \to C_f \to \Sigma X.$$

TR3 Given a solid diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, dashed] & \Sigma X \arrow[d, "\Sigma f"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X'
\end{tikzcd}$$

such that the leftmost square commmutes and both rows are distinguished, there exists a dashed arrow $Z \to Z'$ which makes the remaining two squares commute.

TR4 A triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

is distinguished if and only if

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is distinguished.

TR5 (Octahedral axiom) Given three distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{h} Y/X \to \Sigma X$$

$$Y \xrightarrow{g} Z \xrightarrow{k} Z/Y \to \Sigma Y$$

$$X \xrightarrow{g \circ f} Z \xrightarrow{l} Z/X \to \Sigma X$$

there exists a distinguished triangle

$$Y/X \xrightarrow{u} Z/X \xrightarrow{v} Z/Y \xrightarrow{w} \Sigma(Y/X)$$

such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[rr, "g \circ f"] \arrow[dr, "f"'] & & Z \arrow[rr, "k"] \arrow[dr, "l"'] & & Z/Y \arrow[rr, "w"] \arrow[dr] & & \Sigma(Y/X) \\
& Y \arrow[ur, "g"'] \arrow[dr, "h"'] & & Z/X \arrow[ur, "v"'] \arrow[dr] & & \Sigma Y \arrow[ur, "\Sigma h"'] & \\
& & Y/X \arrow[ur, "u"'] \arrow[rr] & & \Sigma X \arrow[ur, "\Sigma f"'] & &
\end{tikzcd}$$

It turns out that the above definition is actually redundant; TR3 and TR4 follow from the remaining axioms (see Lemmas 2.2 and 2.4 in [18]). In Appendix A, we develop some of the theory of triangulated categories. For those familiar with the theory of model categories, the homotopy category of any stable model category is canonically triangulated (see [12, Chapter 7]). The most commonly considered example of a triangulated category is the derived category $\mathcal{D}(\mathcal{A})$ of an abelian group $\mathcal{A}$, obtained by localizing the category of chain complexes in $\mathcal{A}$ at the quasi-isomorphisms.

In nature, one will often encounter categories which are both triangulated and symmetric monoidal. It is natural to ask that the two structures are compatible in some sense. Such categories are called *tensor triangulated* categories, and there are multiple proposed definitions given in the literature for what these categories look like. For our purposes, we will use Definition 2.1 from Balmer's paper [3], which defines a tensor triangulated category to be a triangulated symmetric monoidal category for which the functor $-\otimes-$ is triangulated in each argument. Unravelling definitions, we may give the following more explicit definition:

Definition 2.2. A *tensor triangulated category* is a triangulated symmetric monoidal category $(\mathcal{C}, \otimes, S, \Sigma, \mathcal{D})$ such that:

TT1 For all objects X and Y in $\mathcal{C}$, there are natural isomorphisms

$$e_{X,Y} : \Sigma X \otimes Y \xrightarrow{\cong} \Sigma(X \otimes Y).$$

TT2 For each object X in $\mathcal{C}$, the functor $X \otimes (-) \cong (-) \otimes X$ is an additive functor.

TT3 For each object X in $\mathcal{C}$, the functor $X \otimes (-) \cong (-) \otimes X$ preserves distinguished triangles, in that given a distinguished triangle/(co)fiber sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A,$$

then also

$$X \otimes A \xrightarrow{X\otimes f} X \otimes B \xrightarrow{X \otimes g} X \otimes C \xrightarrow{X\otimes' h} \Sigma(X \otimes A)$$

and

$$A \otimes X \xrightarrow{f\otimes X} B \otimes X \xrightarrow{g \otimes X} C \otimes X \xrightarrow{h\otimes' X} \Sigma(A \otimes X)$$

are distinguished triangles, where here we writing $X \otimes' h$ and $h \otimes' X$ to denote the compositions

$$X \otimes C \xrightarrow{X\otimes h} X \otimes \Sigma A \xrightarrow{\tau} \Sigma A \otimes X \xrightarrow{e_{A,X}} \Sigma(A \otimes X) \xrightarrow{\Sigma\tau} \Sigma(X \otimes A)$$

and

$$C \otimes X \xrightarrow{h\otimes X} \Sigma A \otimes X \xrightarrow{e_{A,X}} \Sigma(A \otimes X),$$

respectively.

This definition will suffice for our purposes, but we warn the reader that it is the weakest found in the literature. Often additional coherence axioms are imposed, for example, one may require that the $e_{X,Y}$'s to be compatible with the associators and to satisfy a sort of "graded commutativity condition". For an in-depth discussion of such extra conditions, we refer the reader to the treatment given by May in [18]. For examples of tensor triangulated categories, we refer the reader to Section 1 of Balmer's paper [4].

Definition 2.3. Given a tensor triangulated category $(\mathcal{C}, \otimes, S, \Sigma, e, \mathcal{D})$, a *sub-Picard grading* on $\mathcal{C}$ is the following data:

- A pointed abelian group $(A, \mathbf{1})$ along with a homomorphism of pointed groups $h : (A, \mathbf{1}) \to (\operatorname{Pic}\mathcal{C}, \Sigma S)$, where $\operatorname{Pic}\mathcal{C}$ is the *Picard group* of isomorphism classes of invertible objects in $\mathcal{C}$.[2]
- For each $a \in A$, a chosen representative S^a called the *a-sphere* in the isomorphism class $h(a)$. We additionally require $S^0 = S$.
- For each $a, b \in A$, an isomorphism $\phi_{a,b} : S^{a+b} \xrightarrow{\cong} S^a \otimes S^b$. This family of isomorphisms is required to be *coherent*, in the following sense:
  - For all $a \in A$, we must have that $\phi_{a,0}$ coincides with the right unitor $\rho^{-1}_{S^a} : S^a \xrightarrow{\cong} S^a \otimes S$ and $\phi_{0,a}$ coincides the left unitor $\lambda^{-1}_{S^a} : S^a \xrightarrow{\cong} S \otimes S^a$.

[2] Recall an object X is a symmetric monoidal category is *invertible* if there exists some object Y and an isomorphism $S \cong X \otimes Y$.

– For all $a, b, c \in A$, the following "associativity diagram" must commute:

$$\begin{tikzcd} S^{a+b} \otimes S^c \arrow[d, "\phi_{a,b} \otimes S^c"'] & S^{a+b+c} \arrow[l, "\phi_{a+b,c}"'] \arrow[r, "\phi_{a,b+c}"] & S^a \otimes S^{b+c} \arrow[d, "S^a \otimes \phi_{b,c}"] \\ (S^a \otimes S^b) \otimes S^c \arrow[rr, "\cong"] & & S^a \otimes (S^b \otimes S^c) \end{tikzcd}$$

Arguably the most interesting part of the above definition is the family of isomorphisms $\phi_{a,b} : S^{a+b} \xrightarrow{\cong} S^a \otimes S^b$. First of all, note that the two conditions we have given above imply a rather strong notion of coherence for these isomorphisms:

Remark 2.4. By induction, the coherence conditions for the $\phi_{a,b}$'s in the above definition say that given any $a_1, \dots, a_n \in A$ and $b_1, \dots, b_m \in A$ such that $a_1 + \cdots + a_n = b_1 + \cdots + b_m$ and any fixed parenthesizations of $X = S^{a_1} \otimes \cdots \otimes S^{a_b}$ and $Y = S^{b_1} \otimes \cdots \otimes S^{b_m}$, there is a *unique* isomorphism $X \to Y$ that can be obtained by forming formal compositions of products of $\phi_{a,b}$, identities, associators, unitors, and their inverses (but not symmetries).

In light of this remark, when working in a triangulated category with sub-Picard grading, we will usually simply write ϕ or even just $\cong$ for any isomorphism that is built by taking compositions of products of $\phi_{a,b}$'s, unitors, associators, identities, and their inverses.

In [7], Dugger studied the more general notion of an additive symmetric monoidal category $(\mathcal{C}, \otimes, S)$ equipped with an abelian group A and a group homomorphism $h : A \to \mathrm{Pic}(\mathcal{C})$. In particular, there the following question was explored: Given a chosen representative S^a in each isomorphism class $h(a)$ with $S^0 = S$, can one find such a coherent family of isomorphisms $\phi_{a,b} : S^{a+b} \xrightarrow{\cong} S^a \otimes S^b$? (Dugger calls these families "A-trivializations of $\mathcal{C}$"). The answer, given in Proposition 7.1 in Dugger's paper, is that we can always find such a coherent family, although it is certainly not unique, nor is there a canonical choice for such a family. Furthermore, given such a coherent family of isomorphisms, if we define $\pi_*(S)$ to be the A-graded abelian group $\pi_*(S) := \bigoplus_{a \in A} [S^a, S]$, we may endow it with an associative and unital graded product sending $x : S^a \to S$ and $y : S^b \to S$ to the composition

$$S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x \otimes y} S \otimes S \xrightarrow{\cong} S.$$

The bad news is that this product is very much dependent on which choice of coherent family odf isomorphism we chose, and in fact, different coherent families may give rise to strictly non-isomorphic ring structures on $\pi_*(S)$.

The upshot of this discussion is the following: given a tensor triangulated category, in order to give it a sub-Picard grading, all one needs to do is give the information specified in the first two bullet points in Definition 2.3, and then one gets a coherent family of isomorphisms for free, although they must make a choice between several different and non-canonical choices of such families. As we will see in Section 4, this ring structure on $\pi_*(S)$ directly controls a large amount of the additional algebraic structure we can place on hom-groups of objects in $\mathcal{SH}$, so one must be very careful to choose the "correct" family.

2.3. The category $\mathcal{SH}$ and its conventions.

Now, we may finally fix the category $\mathcal{SH}$ in which we will work for the remainder of this document. Before we can do so, we need one last technical definition.

Definition 2.5. Let $\mathcal{C}$ be an additive category with arbitrary (set-indexed) coproducts. Then an object X in $\mathcal{C}$ is *compact* if, for any collection of objects Y_i in $\mathcal{C}$ indexed by some set I, the canonical map

$$\bigoplus_i \mathcal{C}(X, Y_i) \to \mathcal{C}(X, \bigoplus_i Y_i)$$

is an isomorphism of abelian groups. (Explicitly, the above map takes a generator $x \in \mathcal{C}(X, Y_i)$ to the composition $X \xrightarrow{x} Y_i \hookrightarrow \bigoplus_i Y_i$.)

Now we may define the category.

Convention 2.6. Let $(\mathcal{SH}, \otimes, S, \Sigma, e, \mathcal{D})$ be a tensor triangulated category with sub-Picard grading $(A, \mathbf{1}, h, \{S^a\}, \{\phi_{a,b}\})$. We require in addition that:

- $\mathcal{SH}$ is monoidal closed,
- $\mathcal{SH}$ has arbitrary products and coproducts, and
- for each $a \in A$, S^a is a compact object.

The motivating examples of such a category are the following:

- The classical stable homotopy category **hoSp**, which is equipped with an isomorphism
$$h : \mathbb{Z} \xrightarrow{\cong} \mathrm{Pic}(\mathbf{hoSp})$$
sending $n \in \mathbb{Z}$ to the n-sphere spectrum S^n.
- The motivic stable homotopy category $\mathbf{SH}_{\mathscr{S}}$ over a base scheme $\mathscr{S}$, which is equipped with a homomorphism
$$h : \mathbb{Z}^2 \to \mathrm{Pic}(\mathbf{SH}_{\mathscr{S}})$$
sending a pair (p, q) to the motivic (p, q)-sphere spectrum $S^{p,q}$.
- The equivariant stable homotopy category $\mathbf{ho}G\mathbf{Sp}$ associated to a group G, which is equipped with a homomorphism
$$h : RO(G) \to \mathrm{Pic}(\mathbf{ho}G\text{-}\mathbf{Sp})$$
taking a representation V to the representation sphere S^V.

Each of these categories may be realized as the homotopy category of some monoidal stable model category. For a discussion of the classical stable homotopy category and its properties, we refer the reader to the nLab page [22], which gives the construction in explicit detail and proves all the required properties. In particular, we point out the . For the motivic stable homotopy category, we refer the reader to the wonderful treatment given in Section 2 of the paper [30] by Wilson and Østvær. There the construction and properties are only reviewed, and no proofs are given, but at the beginning of the section the authors include a comprehensive list of resources which contain full proofs of all the relevant details. For the equivariant stable homotopy category, we refer the reader to the the book [17] of Mandell and May. We will discuss how exactly the family of $\phi_{a,b}$'s are chosen in these three examples in section 4.

For our purposes, we will not actually need the full power of a closed monoidal structure on $\mathcal{SH}$ — all we will need is that the monoidal product $- \otimes -$ preserves arbitrary (co)limits in each argument. In practice though, and for all the examples we will discuss here, any such category will usually be monoidal closed, so we keep this assumption. In order to reinforce our idea of $\mathcal{SH}$ as "a stable homotopy category", we will establish some relevant notational conventions in $\mathcal{SH}$. Given an object X and a natural number $n > 0$, we write

$$X^n := \overbrace{X \otimes \cdots \otimes X}^{n \text{ times}} \qquad \text{and} \qquad X^0 := S.$$

When we want to be explicit about them, we will denote the associator, symmetry, left unitor, and right unitor isomorphisms in $\mathcal{SH}$ by

$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z) \qquad \tau_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$$
$$\lambda_X : S \otimes X \xrightarrow{\cong} X \qquad \rho_X : X \otimes S \xrightarrow{\cong} X.$$

Often we will drop the subscripts. As we discussed above, by the coherence theorem for symmetric monoidal categories, we will nearly always assume α, ρ, and λ are actual equalities, and will suppress them from the notation entirely.

Given some integer $n \in \mathbb{Z}$, we will write a bold $\mathbf{n}$ to denote the element $n \cdot \mathbf{1}$ in A. Note that given some fixed choice of isomorphism $\gamma : \Sigma S \xrightarrow{\cong} S^{\mathbf{1}}$, we may use it to construct a natural isomorphism $\Sigma \cong S^{\mathbf{1}} \otimes -$:

$$\Sigma X \xrightarrow{\Sigma\lambda_X^{-1}} \Sigma(S \otimes X) \xrightarrow{e_{S,X}^{-1}} \Sigma S \otimes X \xrightarrow{\gamma \otimes X} S^{\mathbf{1}} \otimes X,$$

where $e_{X,Y} : \Sigma X \otimes Y \to \Sigma(X \otimes Y)$ is the isomorphism specified by the fact that $\mathcal{SH}$ is tensor-triangulated. The first two arrows are natural in X by definition. The last arrow is natural in X by functoriality of $- \otimes -$. Henceforth, we will assume some $\gamma : \Sigma S \xrightarrow{\cong} S^1$ has been fixed, and we always use ν to denote the induced natural isomorphism.

Given some $a \in A$, we define functors $\Sigma^a := S^a \otimes -$ and $\Omega^a := \Sigma^{-a} = S^{-a} \otimes -$. We specifically define $\Omega := \Omega^{\mathbf{1}}$. We say "the a^{th} suspension of X" to denote $\Sigma^a X$. It turns out that Σ^a is an autoequivalence of $\mathcal{SH}$ for each $a \in A$, and furthermore, Ω^a and Σ^a form an adjoint equivalence of $\mathcal{SH}$ for all a in A:

Proposition 2.7. *For each $a \in A$, the isomorphisms*

$$\eta_X^a : X \xrightarrow{\phi_{a,-a} \otimes X} S^a \otimes S^{-a} \otimes X = \Sigma^a \Omega^a X$$

and

$$\varepsilon_X^a : \Omega^a \Sigma^a X = S^{-a} \otimes S^a \otimes X \xrightarrow{\phi_{-a,a}^{-1} \otimes X} X$$

are natural in X, and furthermore, they are the unit and counit respectively of the adjoint autoequivalence $(\Omega^a, \Sigma^a, \eta^a, \varepsilon^a)$ of $\mathcal{SH}$.

Proof. That η^a and ε^a are natural in X follows by functoriality of $- \otimes -$. Now, recall that in order to show that these natural isomorphisms form an *adjoint* equivalence, it suffices to show that the natural isomorphisms $\eta^a : \mathrm{Id}_{\mathcal{SH}} \Rightarrow \Omega^a \Sigma^a$ and $\varepsilon^a : \Sigma^a \Omega^a \Rightarrow \mathrm{Id}_{\mathcal{SH}}$ satisfy one of the two zig-zag identities:

$$\begin{tikzcd} \Omega^a \arrow[r, "\Omega^a \eta^a"] \arrow[dr, equal] & \Omega^a \Sigma^a \Omega^a \arrow[d, "\varepsilon^a \Omega^a"] \\ & \Omega^a \end{tikzcd} \qquad \begin{tikzcd} \Sigma^a \Omega^a \Sigma^a \arrow[d, "\Sigma^a \varepsilon^a"'] & \Sigma^a \arrow[l, "\eta^a \Sigma^a"'] \arrow[dl, equal] \\ \Sigma^a & \end{tikzcd}$$

(that it suffices to show only one is [20, Lemma 3.2]). We will show that the left is satisfied. Unravelling definitions, we simply wish to show that the following diagram commutes for all X in $\mathcal{SH}$:

$$\begin{tikzcd} S^{-a} \otimes X \arrow[r, "S^{-a} \otimes \phi_{a,-a} \otimes X"] \arrow[dr, equal] & S^{-a} \otimes S^a \otimes S^{-a} \otimes X \arrow[d, "\phi_{-a,a}^{-1} \otimes S^{-a} \otimes X"] \\ & S^{-a} \otimes X \end{tikzcd}$$

Yet this is simply the diagram obtained by applying $- \otimes X$ to the associativity coherence diagram for the $\phi_{a,b}$'s (since $\phi_{a,0}$ and $\phi_{0,a}$ coincide with the unitors, and by coherence we are taking the unitors and associators to be equalities), so it does commute, as desired. $\square$

In particular, since the functor Σ is naturally isomorphic to $\Sigma^{\mathbf{1}}$, and $\Omega = \Omega^{\mathbf{1}}$ is a left adjoint for Σ, we have that Σ is apart of an adjoint autoequivalence $(\Omega, \Sigma, \eta, \varepsilon)$ of $\mathcal{SH}$, where η and ε are the compositions

$$\eta : \mathrm{Id}_{\mathcal{SH}} \xRightarrow{\eta^{\mathbf{1}}} \Sigma^{\mathbf{1}} \Omega \xRightarrow{\nu^{-1}\Omega} \Sigma\Omega \qquad \text{and} \qquad \varepsilon : \Omega\Sigma \xRightarrow{\Omega\nu} \Omega\Sigma^{\mathbf{1}} \xRightarrow{\varepsilon^{\mathbf{1}}} \mathrm{Id}_{\mathcal{SH}}.$$

In other words, we have shown that the category $\mathcal{SH}$ is *adjointly triangulated*, in the following sense:

Definition 2.8. An *adjointly triangulated category* $(\mathcal{C}, \Omega, \Sigma, \eta, \varepsilon, \mathcal{D})$ is the data of a triangulated category $(\mathcal{C}, \Sigma, \mathcal{D})$ along with an inverse shift functor $\Omega : \mathcal{C} \to \mathcal{C}$ and natural isomorphisms $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow \Sigma\Omega$ and $\varepsilon : \Omega\Sigma \Rightarrow \mathrm{Id}_{\mathcal{C}}$ such that $(\Omega, \Sigma, \eta, \varepsilon)$ forms an adjoint equivalence of $\mathcal{C}$. In other words, η and ε are natural isomorphisms which also are the unit and counit of an adjunction $\Omega \dashv \Sigma$, so they satisfy either of the following "zig-zag identities":

$$\begin{tikzcd} \Omega \arrow[r, "\Omega\eta"] \arrow[dr, equal] & \Omega\Sigma\Omega \arrow[d, "\varepsilon\Omega"] \\ & \Omega \end{tikzcd} \qquad \begin{tikzcd} \Sigma\Omega\Sigma \arrow[d, "\Sigma\varepsilon"'] & \Sigma \arrow[l, "\eta\Sigma"'] \arrow[dl, equal] \\ \Sigma & \end{tikzcd}$$

(Satisfying one implies the other is automatically satisfied, see [20, Lemma 3.2]).

We warn the reader that the above terminology is nonstandard. We prove some results about adjointly triangulated categories in Appendix A.3. Now given two objects X and Y in $\mathcal{SH}$, we will write $[X, Y]$ with brackets to denote the hom-abelian group of morphisms from X to Y, and we will denote the internal hom object by $F(X, Y)$. Keeping with our intuition that $\mathcal{SH}$ is a "homotopy category", we will often refer to elements of $[X, Y]$ as "classes". We may extend the abelian group $[X, Y]$ to an A-graded abelian group $[X, Y]_*$ by defining $[X, Y]_a := [\Sigma^a X, Y]$. It is further possible to extend composition in $\mathcal{SH}$ to an A-graded map

$$[Y, Z]_* \otimes_{\mathbb{Z}} [X, Y]_* \to [X, Z]_*,$$

but we do not explore this here. Given an object X in $\mathcal{SH}$ and some $a \in A$, we can define the abelian group

$$\pi_a(X) := [S^a, X],$$

which we call the a^{th} *(stable) homotopy group of* X. We write $\pi_*(X)$ for the A-graded abelian group $\bigoplus_{a \in A} \pi_a(X)$, so that in particular we have a canonical isomorphism

$$\pi_*(X) = [S^*, X] \cong [S, X]_*.$$

Given some other object E, we can define the A-graded abelian groups $E_*(X)$ and $E^*(X)$ by the formulas

$$E_a(X) := \pi_a(E \otimes X) = [S^a, E \otimes X] \qquad \text{and} \qquad E^a(X) := [X, S^a \otimes E].$$

We refer to the functor $E_*(-)$ as the *homology theory represented by* E, or just E-homology, and we refer to $E^*(-)$ as the *cohomology theory represented by* E, or just E-cohomology.

A nice result is that in $\mathcal{SH}$, (co)fiber sequences (distinguished triangles) give rise to homotopy long exact sequences. Of key importance for this exact sequence (any many applications beyond), will be some fixed family of isomorphisms $s^a_{X,Y} : [X, \Sigma^a Y]_* \xrightarrow{\cong} [X, Y]_{*-a}$. We fix these now, once and for all:

Definition 2.9. For all X, Y in $\mathcal{SH}$ and $a \in A$, there are A-graded isomorphisms

$$s^a_{X,Y} : [X, \Sigma^a Y]_* \to [X, Y]_{*-a}$$

sending $x : S^b \otimes X \to S^a \otimes Y$ in $[X, \Sigma^a Y]_*$ to the composition

$$S^{b-a} \otimes X \xrightarrow{\phi_{-a,b} \otimes X} S^{-a} \otimes S^b \otimes X \xrightarrow{S^{-a} \otimes x} S^{-a} \otimes S^a \otimes Y \xrightarrow{\phi^{-1}_{-a,a} \otimes Y} Y.$$

Furthermore, these isomorphisms are natural in both X and Y.

In particular, for each $a \in A$ and object X in $\mathcal{SH}$, we have natural isomorphisms

$$s^a_X : \pi_*(\Sigma^a X) = [S^*, \Sigma^a X] \xrightarrow{\cong} [S, \Sigma^a X]_* \xrightarrow{s^a_{S,X}} [S, X]_{*-a} \xrightarrow{\cong} \pi_{*-a}(X)$$

sending $x : S^b \to S^a \otimes X$ in $\pi_*(\Sigma^a X)$ to the composition

$$S^{b-a} \xrightarrow{\phi_{-a,b}} S^{-a} \otimes S^b \xrightarrow{S^{-a}\otimes x} S^{-a} \otimes S^a \otimes X \xrightarrow{\phi^{-1}_{-a,a}\otimes X} X.$$

Proof. First, by unravelling definitions, note that $s^a_{X,Y}$ is precisely the composition

$$[X, \Sigma^a Y]_* = [S^* \otimes X, S^a \otimes Y] \xrightarrow{\text{adj}} [S^{-a} \otimes S^* \otimes X, Y] \xrightarrow{(\phi_{-a,*}\otimes X)^*} [S^{*-a} \otimes X, Y] = [X, Y]_{*-a},$$

where the adjunction is that from Proposition 2.7. The adjunction is natural in $S^* \otimes X$ and Y by definition, so that in particular it is natural in X and Y. It is furthermore straightforward to see by functoriality of $- \otimes -$ that the second arrow is natural in both X and Y. Thus $s^a_{X,Y}$ is natural in X and Y, as desired. □

Now we may construct the long exact sequence:

Proposition 2.10. *Suppose we are given a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

and an object W in $\mathcal{SH}$. Then there exists a "connecting homomorphism" of degree $-\mathbf{1}$

$$\partial : [W, Z]_* \to [W, X]_{*-\mathbf{1}}$$

such that the following triangle is exact at each vertex:

$$\begin{tikzcd} {[W,X]_*} \arrow[r, "f_*"] & {[W,Y]_*} \arrow[d, "g_*"] \\ & {[W,Z]_*} \arrow[ul, "\partial"] \end{tikzcd}$$

Proof. By axiom TR4 for a triangulated category and the fact that distinguished triangles are exact (Proposition A.2), we have the following exact sequence in $\mathcal{SH}$

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{\Sigma f} \Sigma Y.$$

Thus, we may apply $[W, -]_*$ to get an exact sequence of A-graded abelian groups which fits into the top row in the following diagram:

$$\begin{tikzcd} {[W,X]_*} \arrow[r, "f_*"] \arrow[dd, equal] & {[W,Y]_*} \arrow[r, "g_*"] \arrow[dd, equal] & {[W,Z]_*} \arrow[r, "h_*"] \arrow[dd, equal] & {[W,\Sigma X]_*} \arrow[r, "\Sigma f_*"] \arrow[d, "(\nu_X)_*"] & {[W,\Sigma Y]_*} \arrow[d, "(\nu_Y)_*"] \\ & & & {[W,\Sigma^{\mathbf{1}} X]_*} \arrow[r, "\Sigma^{\mathbf{1}} f_*"] \arrow[d, "s^{\mathbf{1}}_{W,X}"] & {[W,\Sigma^{\mathbf{1}} Y]_*} \arrow[d, "s^{\mathbf{1}}_{W,Y}"] \\ {[W,X]_*} \arrow[r, "f_*"] & {[W,Y]_*} \arrow[r, "g_*"] & {[W,Z]_*} \arrow[r, dashed, "\partial"] & {[W,X]_{*-\mathbf{1}}} \arrow[r, "f_*"] & {[W,Y]_{*-\mathbf{1}}} \end{tikzcd}$$

where here we define $\partial : [W, Z]_* \to [W, X]_{*-\mathbf{1}}$ to be the composition which makes the third square commute. The diagram commutes by naturality of ν and $s^{\mathbf{1}}$, so that the bottom row is exact since the top row is exact and the vertical arrows are isomorphisms. Thus the bottom row is the long exact sequence, and we may roll it up to get the desired exact triangle:

$$\begin{tikzcd} {[W,X]_*} \arrow[r, "f_*"] & {[W,Y]_*} \arrow[d, "g_*"] \\ & {[W,Z]_*} \arrow[ul, "\partial"] \end{tikzcd}$$

□

3. Cellular objects in $\mathcal{SH}$

One very important class of objects in $\mathcal{SH}$ are the *cellular* objects. Intuitively, these are the objects that can be built out of spheres via taking coproducts and (co)fibers.

Definition 3.1. Define the class of *cellular* objects in $\mathcal{SH}$ to be the smallest class of objects such that:

(1) For all $a \in A$, the a-sphere S^a is cellular.
(2) If we have a distinguished triangle
$$X \to Y \to Z \to \Sigma X$$
such that two of the three objects X, Y, and Z are cellular, than the third object is also cellular.
(3) Given a collection of cellular objects X_i indexed by some (small) set I, the object $\bigoplus_{i \in I} X_i$ is cellular (recall we have chosen $\mathcal{SH}$ to have arbitrary coproducts).

We write $\mathcal{SH}$-**Cell** to denote the full subcategory of $\mathcal{SH}$ on the cellular objects.

This definition is adapted from that given on pg. 21 in the paper [9] by Dugger and Isaksen. By the same reasoning given therewithin, in the motivic stable homotopy category, our definition is equivalent to the original definition of cellularity given in [8]. More generally, given an object E in $\mathcal{SH}$, Dugger and Isaksen consider the class of *E-cellular objects* in $\mathcal{SH}$. This is defined to be the smallest class of objects in $\mathcal{SH}$ satisfying the above three conditions, and in addition the requirement that if X is E-cellular then so is $E \otimes X$. Note that by Lemma 3.4, the class of cellular objects in $\mathcal{SH}$ is equivalently the class of S-cellular objects in $\mathcal{SH}$. For our purposes, we will only be concerned with standard cellular objects, so we will not pursue any of the theory of E-cellular objects in $\mathcal{SH}$ for a general E. For an extensive review of cellularity in the motivic stable homotopy category, as well as a treatment of its construction, we refer the reader to the delightful paper [11] written by Joseph Hlavinka for the 2021 UChicago mathematics REU.

We devote the rest of the section to proving some important facts about cellular objects. These should be familiar to anyone acquainted with the usual notion of cellular spaces (CW complexes).

Lemma 3.2. *Let X and Y be two isomorphic objects in $\mathcal{SH}$. Then X is cellular iff Y is cellular.*

Proof. Assume we have an isomorphism $f : X \xrightarrow{\cong} Y$ and that X is cellular. Then consider the following commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r] \arrow[d, "f^{-1}"] & 0 \arrow[r] \arrow[d, equal] & \Sigma X \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[r] & 0 \arrow[r] & \Sigma X
\end{tikzcd}$$

The bottom row is distinguished by axiom TR1 for a triangulated category. Hence since X is cellular, 0 is also cellular, since the class of cellular objects satisfies two-of-three for distinguished triangles. Furthermore, since the vertical arrows are all isomorphisms, the top row is distinguished as well, by axiom TR0. Thus again by two-of-three, since X and 0 are cellular, so is Y, as desired. $\square$

Example 3.3. Every object in the classical stable homotopy category is cellular.

Proof. By [23, Proposition 2.16], every object in **hoSp** is isomorphic to a CW spectrum, which are spectra that can be constructed by gluing copies of spheres together, and are thus clearly cellular (since the cofiber of $f : X \to Y$ in **hoSp** is precisely the spectrum obtained by “gluing” a disk to Y along $f(X)$). $\square$

Lemma 3.4. *Let X and Y be cellular objects in $\mathcal{SH}$. Then $X \otimes Y$ is cellular.*

Proof. Let E be a cellular object in $\mathcal{SH}$, and let $\mathcal{E}$ be the collection of objects X in $\mathcal{SH}$ such that $E \otimes X$ is cellular. First of all, suppose we have a distinguished triangle

$$X \to Y \to Z \to \Sigma X$$

such that two of three of X, Y, and Z belong to $\mathcal{E}$. Then since $\mathcal{SH}$ is tensor triangulated, we have a distinguished triangle

$$E \otimes X \to E \otimes Y \to E \otimes Z \to \Sigma(E \otimes X).$$

Per our assumptions, two of three of $E \otimes X$, $E \otimes Y$, and $E \otimes Z$ are cellular, so that the third is by definition. Thus, all three of X, Y, and Z belong to $\mathcal{E}$ if two of them do.

Second of all, suppose we have a family X_i of objects in $\mathcal{E}$ indexed by some (small) set I, and set $X := \bigoplus_i X_i$. Then we'd like to show X belongs to $\mathcal{E}$, i.e., that $E \otimes X$ is cellular. Indeed,

$$E \otimes X = E \otimes \left(\bigoplus_i X_i \right) \cong \bigoplus_i (E \otimes X_i),$$

where the isomorphism is given by the fact that $\mathcal{SH}$ is monoidal closed, so $E \otimes -$ preserves arbitrary colimits as it is a left adjoint. Per our assumption, since each $E \otimes X_i$ is cellular, the rightmost object is cellular, since the class of cellular objects is closed under taking arbitrary coproducts, by definition. Hence $E \otimes X$ is cellular by Lemma 3.2.

Finally, we would like to show that each S^a belongs to $\mathcal{E}$, i.e., that $S^a \otimes E$ is cellular for all $a \in A$. When $E = S^b$ for some $b \in A$, this is clearly true, since $S^b \otimes S^a \cong S^{a+b}$, which is cellular by definition, so that $S^b \otimes S^a$ is cellular by Lemma 3.2. Thus by what we have shown, the class of objects X for which $S^a \otimes X$ is cellular contains every cellular object. Hence in particular $E \otimes S^a \cong S^a \otimes E$ is cellular for all $a \in A$, as desired. $\square$

Lemma 3.5. *Let W be a cellular object in $\mathcal{SH}$ such that $\pi_*(W) = 0$. Then $W \cong 0$.*

Proof. Let $\mathcal{E}$ be the collection of all X in $\mathcal{SH}$ such that and $[\Sigma^n X, W] = 0$ for all $n \in \mathbb{Z}$ (where for $n > 0$ we define $\Sigma^{-n} := \Omega^n = (S^{-\mathbf{1}} \otimes -)^n$). We claim $\mathcal{E}$ contains every cellular object in $\mathcal{SH}$. First of all, each S^a belongs to $\mathcal{E}$, as

$$[\Sigma^n S^a, W] \cong [S^{\mathbf{n}} \otimes S^a, W] \cong [S^{a+\mathbf{n}}, W] \leq \pi_*(W) = 0.$$

Furthermore, suppose we are given a distinguished triangle

$$X \to Y \to Z \to \Sigma X$$

such that two of three of X, Y, and Z belong to $\mathcal{E}$. By Proposition A.9, for all $n \in \mathbb{Z}$ we get an exact sequence of abelian groups

$$[\Sigma^{n+1} X, W] \to [\Sigma^n Z, W] \to [\Sigma^n Y, W] \to [\Sigma^n X, W] \to [\Sigma^{n-1} Z, W].$$

Clearly if any two of three of X, Y, and Z belong to $\mathcal{E}$, then by exactness of the above sequence all three of the middle terms will be zero, so that the third object will belong to $\mathcal{E}$ as well. Finally, suppose we have a collection of objects X_i in $\mathcal{E}$ indexed by some small set I. Then

$$\left[\Sigma^n \bigoplus_i X_i, W \right] \cong \left[\bigoplus_i \Sigma^n X_i, W \right] \cong \prod_i [\Sigma^n X_i, W] = \prod_i 0 = 0,$$

where the first isomorphism follows by the fact that Σ^n is apart of an adjoint equivalence (Proposition 2.7), so it preserves arbitrary colimits.

Thus, by definition of cellularity, $\mathcal{E}$ contains every cellular object. In particular, $\mathcal{E}$ contains W, so that $[W, W] = 0$, meaning $\mathrm{id}_W = 0$, so we have a commutative diagram

$$\begin{tikzcd} & 0 \arrow[rr, equal] \arrow[dr] & & 0 \\ W \arrow[ur] \arrow[rr, equal] & & W \arrow[ur] & \end{tikzcd}$$

Hence the diagonals exhibit isomorphisms between 0 and W, as desired. □

Theorem 3.6. *Let X and Y be cellular objects in $\mathcal{SH}$, and suppose $f : X \to Y$ is a morphism such that $f_* : \pi_*(X) \to \pi_*(Y)$ is an isomorphism. Then f is an isomorphism.*

Proof. By axiom TR2 for a triangulated category, we have a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} C_f \xrightarrow{h} \Sigma X.$$

First of all, note that by definition since X and Y are cellular, so is C_f. We claim $\pi_*(C_f) = 0$. Indeed, given $a \in A$, by axiom TR4 for a triangulated category and the fact that distinguished triangles are exact, the following sequence of abelian groups is exact:

$$[S^a, X] \xrightarrow{f_*} [S^a, Y] \xrightarrow{g_*} [S^a, C_f] \xrightarrow{h_*} [S^a, \Sigma X] \xrightarrow{\Sigma f_*} [S^a, \Sigma Y].$$

where the first arrow is and last arrows are isomorphisms, per our assumption that f is an isomorphism. Then by exactness we have $\operatorname{im} h_* = \ker(\Sigma f_*) = 0$. Yet we also have $\ker g_* = \operatorname{im} f_* = [S^a, Y]$, so that $\ker h_* = \operatorname{im} g_* = 0$. It is only possible that $\ker h_* = \operatorname{im} h_* = 0$ if $[S^a, C_f] = 0$. Thus, we have shown $\pi_*(C_f) = 0$, and C_f is cellular, so by Lemma 3.5 there is an isomorphism $C_f \cong 0$. Now consider the following diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"] & Y \arrow[r] \arrow[d, equal] & C_f \arrow[r] \arrow[d, "\cong"] & \Sigma X \arrow[d, "\Sigma f"] \\ Y \arrow[r, equal] & Y \arrow[r] & 0 \arrow[r] & \Sigma Y \end{tikzcd}$$

The middle square commutes since 0 is terminal, while the right square commutes since $C_f \cong 0$ is initial. The top row is distinguished by assumption. The bottom row is distinguished by axiom TR2. Then since the middle two vertical arrows are isomorphisms, by Lemma A.3, f is an isomorphism as well, as desired. □

Lemma 3.7. *Let $e : X \to X$ be an idempotent morphism in $\mathcal{SH}$, i.e., $e \circ e = e$. Then this idempotent splits, meaning e factors as*

$$X \xrightarrow{r} Y \xrightarrow{\iota} X$$

for some object Y and morphisms r and ι with $r \circ \iota = \mathrm{id}_Y$. Furthermore, if X is cellular than so is Y.

Proof. In [19, Proposition 1.6.8], it is shown that idempotents split in triangulated categories with countable coproducts, and in particular, the object Y through which the splitting factors may be taken as the homotopy colimit of the sequence

$$X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} X \to \cdots .$$

Thus since $\mathcal{SH}$ is triangulated and has arbitrary coproducts, given an idempotent $e : X \to X$ in $\mathcal{SH}$, e splits as desired. Furthermore, the splitting factors through the homotopy limit Y of the above sequence, so we have a distinguished triangle in $\mathcal{SH}$

$$\bigoplus_{i=0}^{\infty} X \to \bigoplus_{i=0}^{\infty} X \to Y \to \Sigma(\bigoplus_{i=0}^{\infty} X).$$

Then if X is cellular, by definition $\bigoplus_{i=0}^{\infty} X$ is as well. Thus by 2-of-3 for distinguished triangles for cellular objects, we would have that Y is cellular as desired. $\square$

4. Monoid objects in $\mathcal{SH}$

So far, we have shown that each object E in $\mathcal{SH}$ yields an E-homology functor E_* from $\mathcal{SH}$ to the category $\mathbf{Ab}^A$ of A-graded abelian groups. In this section, we will examine some conditions on E under which we may refine this functor by identifying more structure on its image. The key assumption will be that E is a *monoid object* in $\mathcal{SH}$, i.e., that there is an associative and unital multiplication $\mu : E \otimes E \to E$. For a review of monoid objects in a symmetric monoidal category, see Appendix C. The most important example of a monoid object in $\mathcal{SH}$ is the unit S, which has multiplication map $\phi_{0,0}^{-1} = \lambda_S = \rho_S : S \otimes S \to S$ and unit map $\mathrm{id}_S : S \to S$.

4.1. Monoid objects in $\mathcal{SH}$ and their associated rings.

To start, we will show that if E is a monoid object in $\mathcal{SH}$, then $\pi_*(E)$ is canonically a ring.

Proposition 4.1. *The assignment $(E, \mu, e) \mapsto \pi_*(E)$ is a functor π_* from the category $\mathbf{Mon}_{\mathcal{SH}}$ of monoid objects in $\mathcal{SH}$ (Definition C.3) to the category of A-graded rings. In particular, given a monoid object (E, μ, e) in $\mathcal{SH}$, $\pi_*(E)$ is canonically a ring with unit $e \in \pi_0(e) = [S, E]$ and product $\pi_*(E) \times \pi_*(E) \to \pi_*(E)$ which sends classes $x : S^a \to E$ and $y : S^b \to E$ to the composition*

$$xy : S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes E \xrightarrow{\mu} E.$$

Proof. First, we show that $\pi_*(E)$ is actually a ring as indicated. By Lemma B.8, in order to make the A-graded abelian group $\pi_*(E)$ into an A-graded ring, it suffices to construct an associative, unital, and bilinear (distributive) product only with respect to homogeneous elements. Suppose we have classes x, y, and z in $\pi_a(E)$, $\pi_b(E)$, and $\pi_c(E)$, respectively. To see associativity, consider the following diagram:

$$\begin{tikzcd}
 & & & E \otimes E \arrow[d, "\mu"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^a \otimes S^b \otimes S^c \arrow[r, "x \otimes y \otimes z"] & E \otimes E \otimes E \arrow[ur, "\mu \otimes E"] \arrow[dr, "E \otimes \mu"'] & E \\
 & & & E \otimes E \arrow[u, "\mu"']
\end{tikzcd}$$

(here the first arrow is the unique isomorphism obtained by composing products of $\phi_{a,b}$'s, see Remark 2.4). It commutes by associativity of μ. It follows by functoriality of $- \otimes -$ that the top composition is $(x \cdot y) \cdot z$ while the bottom is $x \cdot (y \cdot z)$, so they are equal as desired. To see that $e \in \pi_0(E)$ is a left and right unit for this multiplication, consider the following diagram

$$\begin{tikzcd}
 & S^a \arrow[dl, "e \otimes x"'] \arrow[d, "x"] \arrow[dr, "x \otimes e"] & \\
E \otimes E \arrow[dr, "\mu"'] & E \arrow[l, "e \otimes E"'] \arrow[r, "E \otimes e"] \arrow[d, equal] & E \otimes E \arrow[dl, "\mu"] \\
 & E &
\end{tikzcd}$$

Commutativity of the two top triangles is functoriality of $- \otimes -$. Commutativity of the bottom two triangles is unitality of μ. Thus the diagram commutes, so $e \cdot x = x = x \cdot e$. Finally, we wish to show this product is bilinear (distributive). Suppose we further have some $x' \in \pi_a(E)$ and

$y' \in \pi_b(E)$, and consider the following diagrams:

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] \arrow[d, "\Delta"'] & S^a \otimes S^b \arrow[r, "\Delta \otimes S^b"] \arrow[d, "\Delta"] & (S^a \oplus S^a) \otimes S^b \arrow[r, "(x \oplus x') \otimes y"] \arrow[dl, "\cong"'] & (E \oplus E) \otimes E \arrow[d, "\nabla \otimes E"] \arrow[dl, "\cong"'] & \\
S^{a+b} \oplus S^{a+b} \arrow[r, "\phi_{a,b} \oplus \phi_{a,b}"'] & (S^a \otimes S^b) \oplus (S^a \otimes S^b) \arrow[r, "(x \otimes y) \oplus (x' \otimes y)"'] & (E \otimes E) \oplus (E \otimes E) \arrow[r, "\nabla"'] & E \otimes E \arrow[r, "\mu"] & E
\end{tikzcd}$$

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] \arrow[d, "\Delta"'] & S^a \otimes S^b \arrow[r, "S^a \otimes \Delta"] \arrow[d, "\Delta"] & S^b \otimes (S^b \oplus S^b) \arrow[r, "x \otimes (y \oplus y')"] \arrow[dl, "\cong"'] & E \otimes (E \oplus E) \arrow[d, "E \otimes \nabla"] \arrow[dl, "\cong"'] & \\
S^{a+b} \oplus S^{a+b} \arrow[r, "\phi_{a,b} \oplus \phi_{a,b}"'] & (S^a \otimes S^b) \oplus (S^a \otimes S^b) \arrow[r, "(x \otimes y) \oplus (x \otimes y')"'] & (E \otimes E) \oplus (E \otimes E) \arrow[r, "\nabla"'] & E \otimes E \arrow[r, "\mu"] & E
\end{tikzcd}$$

The unlabeled isomorphisms are those given by the fact that $- \otimes -$ is additive in each variable (since $\mathcal{SH}$ is tensor triangulated). Commutativity of the left squares is naturality of $\Delta : X \to X \oplus X$ in an additive category. Commutativity of the rest of the diagram follows again from the fact that $- \otimes -$ is an additive functor in each variable. Hence, by functoriality of $- \otimes -$, these diagrams tell us that $(x + x') \cdot y = x \cdot y + x' \cdot y$ and $x \cdot (y + y') = x \cdot y + x \cdot y'$, respectively. Thus, we have shown that if (E, μ, e) is a monoid object in $\mathcal{SH}$ then $\pi_*(E)$ is a ring, as desired.

It remains to show that given a homomorphism of monoid objects $f : (E_1, \mu_1, e_1) \to (E_2, \mu_2, e_2)$ in $\mathbf{Mon}_{\mathcal{SH}}$ that $\pi_*(f) : \pi_*(E_1) \to \pi_*(E_2)$ is an A-graded ring homomorphism. First of all, we know this is an A-graded abelian group homomorphism, since $\mathcal{SH}$ is an additive category, meaning composition with f is an abelian group homomorphism. Thus, in order to show it's a ring homomorphism, it remains to show that $\pi_*(f)(e_1) = e_2$ and that for all $x, y \in \pi_*(E)$ we have $\pi_*(f)(x \cdot y) = \pi_*(f)(x) \cdot \pi_*(f)(y)$. The former follows since $\pi_*(f)(e_1) = f \circ e_1 = e_2$, since f is a monoid homomorphism in $\mathcal{SH}$. To see the latter, first note by distributivity of multiplication in $\pi_*(E_1)$ and $\pi_*(E_2)$ and the fact that $\pi_*(f)$ is a group homomorphism, it suffices to consider the case that x and y are homogeneous of the form $x : S^a \to E_1$ and $y : S^b \to E_2$. In this case, consider the following diagram:

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] & S^a \otimes S^b \arrow[r, "x \otimes y"] & E_1 \otimes E_1 \arrow[r, "f \otimes f"] \arrow[d, "\mu_1"'] & E_2 \otimes E_2 \arrow[d, "\mu_2"] \\
& & E_1 \arrow[r, "f"] & E_2
\end{tikzcd}$$

The top composition is $\pi_*(f)(x) \cdot \pi_*(f)(y)$, while the bottom composition is $\pi_*(f)(x \cdot y)$. The diagram commutes since f is a monoid object homomorphism. Thus $\pi_*(f)(x \cdot y) = \pi_*(f)(x) \cdot \pi_*(f)(y)$, as desired. $\square$

The most important example of such a ring will be the *stable homotopy ring* $\pi_*(S)$, which controls essentially the entire structure of $\mathcal{SH}$. We have shown that π_* takes monoids to rings. Next, we will show that given a monoid object (E, μ, e) in $\mathcal{SH}$, the functor E_* is valued in A-graded left $\pi_*(E)$-modules. First, we prove the following lemma:

Lemma 4.2. *Let X and Y be objects in $\mathcal{SH}$. Then the A-graded pairing*

$$\pi_*(X) \times \pi_*(Y) \to \pi_*(X \otimes Y)$$

sending $x : S^a \to X$ and $y : S^b \to Y$ to the composition

$$S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x \otimes y} X \otimes Y$$

is bilinear, i.e., it is additive in each argument.

Proof. Let $a, b \in A$, and let $x_1, x_2 : S^a \to X$ and $y : S^b \to Y$. Then consider the following diagram

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\cong"] & S^a \otimes S^b \arrow[r, "\Delta \otimes S^b"] \arrow[d, "\Delta"'] & (S^a \oplus S^a) \otimes S^b \arrow[dl, "\cong"'] \arrow[d, "(x_1 \oplus x_2) \otimes y"] \\
 & (S^a \otimes S^b) \oplus (S^a \otimes S^b) \arrow[d, "(x_1 \otimes y) \oplus (x_2 \otimes y)"'] & (X \oplus X) \otimes Y \arrow[dl, "\cong"'] \arrow[d, "\nabla \otimes Y"] \\
 & (X \otimes Y) \oplus (X \otimes Y) \arrow[r, "\nabla"] & X \otimes Y
\end{tikzcd}$$

The isomorphisms are given by the fact that $- \otimes -$ is additive in each variable. Both triangles and the parallelogram commute since $- \otimes -$ is additive. By functoriality of $- \otimes -$, the top composition is $(x_1 + x_2) \cdot y$ and the bottom composition is $x_1 \cdot y + x_2 \cdot y$, so they are equal, as desired. An entirely analagous argument yields that $x \cdot (y_1 + y_2) = x \cdot y_1 + x \cdot y_2$ for $x \in \pi_*(X)$ and $y_1, y_2 \in \pi_*(Y)$. $\square$

Now we can show that $E_*(X)$ is a graded module over $\pi_*(E)$.

Proposition 4.3. *Let (E, μ, e) be a monoid object in $\mathcal{SH}$. Then $E_*(-)$ is an additive functor from $\mathcal{SH}$ to the category $\pi_*(E)$-$\mathbf{Mod}^A$ of left A-graded modules over the ring $\pi_*(E)$ (Proposition 4.1) and degree-preserving homomorphisms between them, where given some X in $\mathcal{SH}$, $E_*(X)$ may be endowed with its* canonical *structure as a left A-graded $\pi_*(E)$-module via the map*

$$\pi_*(E) \times E_*(X) \to E_*(X)$$

which given $a, b \in A$, sends $x : S^a \to E$ and $y : S^b \to E \otimes X$ to the composition

$$x \cdot y : S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes E \otimes X \xrightarrow{\mu \otimes X} E \otimes X.$$

Similarly, the assignment $X \mapsto X_*(E)$ is a functor from $\mathcal{SH}$ to right A-graded $\pi_*(E)$-modules, where the structure map

$$X_*(E) \times \pi_*(E) \to X_*(E)$$

sends $x : S^a \to X \otimes E$ and $y : S^b \to E$ to the composition

$$x \cdot y : S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x \otimes y} X \otimes E \otimes E \xrightarrow{X \otimes \mu} X \otimes E.$$

Finally, $E_*(E)$ is a $\pi_*(E)$-bimodule, in the sense that the left and right actions of $\pi_*(E)$ are compatible, so that given $y, z \in \pi_*(E)$ and $x \in E_*(E)$, $y \cdot (x \cdot z) = (y \cdot x) \cdot z$.

Proof. By Lemma B.9, in order to make the A-graded abelian group $E_*(X)$ into a left A-graded module over the A-graded ring $\pi_*(E)$, it suffices to define the action map $\pi_*(E) \times E_*(X) \to E_*(X)$ only for homogeneous elements, and to show that given homogeneous elements $x, x' : S^a \to E \otimes X$ in $E_a(X)$, $y : S^b \to E$ in $\pi_b(E)$, and $z, z' : S^c \to E$ in $\pi_c(E)$, that:

(1) $y \cdot (x + x') = y \cdot x + y \cdot x'$,
(2) $(z + z') \cdot x = z \cdot x + z' \cdot x$,
(3) $(zy) \cdot x = z \cdot (y \cdot x)$,
(4) $e \cdot x = x$.

Items (1) and (2) follow by the fact that $E_*(X) = \pi_*(E \otimes X)$ and Lemma 4.2. To see (3), consider the diagram:

$$\begin{tikzcd}
 & & & E \otimes E \otimes X \arrow[d, "\mu \otimes X"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^c \otimes S^b \otimes S^a \arrow[r, "z \otimes y \otimes x"] & E \otimes E \otimes E \otimes X \arrow[ur, "E \otimes \mu \otimes X"] \arrow[dr, "\mu \otimes E \otimes X"'] & E \otimes X \\
 & & & E \otimes E \otimes X \arrow[u, "\mu \otimes X"']
\end{tikzcd}$$

It commutes by associativity of μ. By functoriality of $- \otimes -$, the two outside compositions equal $z \cdot (y \cdot x)$ on the top and $(z \cdot y) \cdot x$ on the bottom. Hence, they are equal, as desired. Next, to see (4), consider the following diagram:

$$\begin{tikzcd}
S^a \arrow[rr, "x"] \arrow[dr, "x"'] \arrow[ddr, bend right, "e \otimes x"'] & & E \otimes X \\
 & E \otimes X \arrow[ur, equal] \arrow[d, "e \otimes E \otimes X"] & \\
 & E \otimes E \otimes X \arrow[uur, bend right, "\mu \otimes X"'] &
\end{tikzcd}$$

The top triangle commutes by definition. The left triangle commutes by functoriality of $- \otimes -$. The right triangle commutes by unitality of μ. The top composition is x while the bottom is $e \cdot x$, thus they are necessarily equal since the diagram commutes.

Thus, we have shown that the indicated map does indeed endow $E_*(X)$ with the structure of a left $\pi_*(E)$-module. Next we would like to show that $E_*(-)$ sends maps in $\mathcal{SH}$ to A-graded homomorphisms of left A-graded $\pi_*(E)$-modules. By definition, given $f : X \to Y$ in $\mathcal{SH}$, $E_*(f) = [S^*, E \otimes f]$ is the map which takes a class $x : S^a \to E \otimes X$ to the composition

$$S^a \xrightarrow{x} E \otimes X \xrightarrow{E \otimes f} E \otimes Y.$$

Since $\mathcal{SH}$ is additive, composition is bilinear, so $[S^*, E \otimes f]$ is an A-graded group homomorphism by definition. To see that it is a further a homomorphism of $\pi_*(E)$-modules, it suffices to show that given classes $x : S^a \to E \otimes X$ and $y : S^b \to E$ that $E_*(f)(y \cdot x) = y \cdot E_*(f)(x)$. To that end, consider the following diagram:

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{b,a}"] & S^b \otimes S^a \arrow[r, "y \otimes x"] & E \otimes E \otimes X \arrow[r, "E \otimes E \otimes f"] \arrow[d, "\mu \otimes X"'] & E \otimes E \otimes Y \arrow[d, "\mu \otimes Y"] \\
 & & E \otimes X \arrow[r, "E \otimes f"] & E \otimes Y
\end{tikzcd}$$

It commutes by functoriality of $- \otimes -$. The top composition is $E_*(f)(y \cdot x)$, while the bottom composition is $y \cdot E_*(f)(x)$, so they are equal, as desired.

Thus, we've shown $E_*(-)$ yields a functor $\mathcal{SH} \to \pi_*(E)\text{-}\mathbf{Mod}^A$; it remains to show this functor is additive, equivalently, **Ab**-enriched. This is clear, as given $f, g : X \to Y$ in $\mathcal{SH}$, we have

$$E_*(f + g) = [S^*, E \otimes (f + g)] = [S^*, (E \otimes f) + (E \otimes g)] = E_*(f) + E_*(g),$$

where the second equality follows since $- \otimes -$ is additive in each variable.

Showing that $X_*(E)$ has the structure of a right $\pi_*(E)$-module and that if $f : X \to Y$ is a morphism in $\mathcal{SH}$ then the map

$$X_*(E) = [S^*, X \otimes E] \xrightarrow{(f \otimes E)_*} [S^*, Y \otimes E] = Y_*(E)$$

is an A-graded homomorphism of right A-graded $\pi_*(E)$-modules is entirely analagous.

It remains to show that $E_*(E)$ is a $\pi_*(E)$-bimodule. Let $x : S^a \to E$, $y : S^b \to E \otimes E$, and $z : S^c \to E$, and consider the following diagram:

$$\begin{tikzcd} & & & E \otimes E \otimes E \arrow[d, "E\otimes\mu"] \\ S^{a+b+c} \arrow[r, "\cong"] & S^a \otimes S^b \otimes S^c \arrow[r, "x\otimes y\otimes z"] & E \otimes E \otimes E \otimes E \arrow[ur, "\mu\otimes E\otimes E"] \arrow[r, "\mu\otimes\mu"] \arrow[dr, "E\otimes E\otimes\mu"'] & E \otimes E \\ & & & E \otimes E \otimes E \arrow[u, "\mu\otimes E"'] \end{tikzcd}$$

Commutativity follows by functoriality of $-\otimes-$, which also tells us that the two outside compositions are $(x \cdot y) \cdot z$ (on top) and $x \cdot (y \cdot z)$ (on bottom). Hence they are equal, as desired. $\square$

Lemma 4.4. *Let E and X be objects in $\mathcal{SH}$. Then for all $a \in A$, there is an A-graded isomorphism of A-graded abelian groups*

$$t^a_X : E_*(\Sigma^a X) \cong E_{*-a}(X)$$

which sends a class $x : S^b \to E \otimes \Sigma^a X = E \otimes S^a \otimes X$ to the composition

$$S^{b-a} \xrightarrow{\phi_{b,-a}} S^b \otimes S^{-a} \xrightarrow{x\otimes S^{-a}} E \otimes S^a \otimes X \otimes S^{-a} \xrightarrow{E\otimes\tau\otimes S^{-a}} E \otimes X \otimes S^a \otimes S^{-a} \xrightarrow{E\otimes X\otimes\phi^{-1}_{a,-a}} E \otimes X$$

with inverse $(t^a_X)^{-1} : E_{*-a}(X) \to E_*(\Sigma^a X)$ sending a class $x : S^{b-a} \to E \otimes X$ to the composition

$$S^b \xrightarrow{\phi_{b-a,a}} S^{b-a} \otimes S^a \xrightarrow{x\otimes S^a} E \otimes X \otimes S^a \xrightarrow{E\otimes\tau} E \otimes S^a \otimes X$$

(where here we are suppressing associators and unitors from the notation). Furthermore this isomorphism is natural in X, and if E is a monoid object in $\mathcal{SH}$ then it is an isomorphism of left $\pi_*(E)$-modules.

Proof. Expressed in terms of hom-sets, t^a_X is precisely the composition

$$\begin{tikzcd} E_*(\Sigma^a X) \arrow[r, equal] & {[S^*, E \otimes S^a \otimes X]} \arrow[d, "(E\otimes\tau)_*"] \\ & {[S^*, E \otimes X \otimes S^a]} \arrow[d, "-\otimes S^{-a}"] \\ & {[S^* \otimes S^{-a}, E \otimes X \otimes S^a \otimes S^{-a}]} \arrow[d, "(E\otimes X\otimes\phi^{-1}_{a,-a})_*"] \\ & {[S^* \otimes S^{-a}, E \otimes X]} \arrow[d, "(\phi_{*,-a})^*"] \\ & {[S^{*-a}, E \otimes X]} \arrow[r, equal] & E_{*-a}(E \otimes X) \end{tikzcd}$$

We know the second vertical arrow is an isomorphism of abelian groups as $-\otimes-$ is additive in each variable (since $\mathcal{SH}$ is tensor triangulated) and $\Omega^a \cong -\otimes S^{-a}$ is an autoequivalence of $\mathcal{SH}$ by Proposition 2.7. The three other vertical arrows are given by composing with an isomorphism in an additive category, so they are also isomorphisms. Now, note the proposed inverse constructed above can be factored into the following composition:

$$\begin{tikzcd}
E_{*-a}(E \otimes X) \arrow[r, equal] & {[S^{*-a}, E \otimes X]} \arrow[d, "- \otimes S^a"] \\
& {[S^{*-a} \otimes S^a, E \otimes X \otimes S^a]} \arrow[d, "(\phi_{*-a,a})^*"] \\
& {[S^*, E \otimes X \otimes S^a]} \arrow[d, "(E \otimes \tau)_*"] \\
& {[S^*, E \otimes S^a \otimes X]} \arrow[r, equal] & E_*(\Sigma^a X)
\end{tikzcd}$$

It is entirely straightforward to check that this is an inverse to t^a_X, and we leave it to the reader to check this. (Since we already know t^a_X is an isomorphism, it suffices to show this composition is either a left or right inverse.)

Now, to see t^a_X is a homomorphism of left $\pi_*(E)$-modules, suppose we are given classes $r : S^b \to E$ in $\pi_b(E)$ and $x : S^c \to E \otimes S^a \otimes X$ in $E_c(\Sigma^a X)$. Then we wish to show that $t^a_X(r \cdot x) = r \cdot t^a_X(x)$. To that end, consider the following diagram:

$$\begin{tikzcd}
S^{b+c-a} \arrow[d, "\cong"] & E \otimes S^a \otimes X \otimes S^{-a} \arrow[r, "E \otimes \tau \otimes S^{-a}"] & E \otimes X \otimes S^a \otimes S^{-a} \arrow[d, "E \otimes X \otimes \phi^{-1}_{a,-a}"] \\
S^b \otimes S^c \otimes S^{-a} \arrow[r, "r \otimes x \otimes S^{-a}"] & E \otimes E \otimes S^a \otimes X \otimes S^{-a} \arrow[u, "\mu \otimes S^a \otimes X \otimes S^{-a}"'] \arrow[d, "E \otimes E \otimes \tau \otimes S^{-a}"'] & E \otimes X \\
& E \otimes E \otimes X \otimes S^a \otimes S^{-a} \arrow[r, "E \otimes E \otimes X \otimes \phi^{-1}_{a,-a}"'] \arrow[uur, "\mu \otimes X \otimes S^a \otimes S^{-a}"] & E \otimes E \otimes X \arrow[u, "\mu \otimes X"']
\end{tikzcd}$$

Both triangles commute by functoriality of $- \otimes -$. The top composition is $t^a_X(r \cdot x)$ while the bottom is $r \cdot t^a_X(x)$, so they are equal as desired.

It remains to show t^a_X is natural in X. let $f : X \to Y$ in $\mathcal{SH}$, then we would like to show the following diagram commutes:

$$\begin{tikzcd}
E_*(\Sigma^a X) \arrow[r, "t^a_X"] \arrow[d, "E_*(\Sigma^a f)"'] & E_{*-a}(X) \arrow[d, "E_{*-a}(f)"] \\
E_*(\Sigma^a Y) \arrow[r, "t^a_Y"] & E_{*-a}(Y)
\end{tikzcd} \tag{1}$$

We may chase a generator around the diagram since all the arrows here are homomorphisms. Let $x : S^b \to E \otimes S^a \otimes X$ in $E_*(\Sigma^a X)$. Then consider the following diagram:

$$\begin{tikzcd}
S^{b-a} \arrow[r, "\cong"] & S^b \otimes S^{-a} \arrow[r, "x \otimes S^{-a}"] & E \otimes S^a \otimes X \otimes S^{-a} \arrow[r, "E \otimes \tau \otimes S^{-a}"] \arrow[d, "E \otimes S^a \otimes f \otimes S^{-a}"'] & E \otimes X \otimes S^a \otimes S^{-a} \arrow[r, "E \otimes X \otimes \phi^{-1}_{a,-a}"] \arrow[d, dashed, "E \otimes f \otimes S^a \otimes S^{-a}"'] & E \otimes X \arrow[d, "E \otimes f"] \\
& & E \otimes S^a \otimes Y \otimes S^{-a} \arrow[r, "E \otimes \tau \otimes S^{-a}"'] & E \otimes Y \otimes S^a \otimes S^{-a} \arrow[r, "E \otimes Y \otimes \phi^{-1}_{a,-a}"'] & E \otimes Y
\end{tikzcd}$$

The left rectangle commutes by naturality of τ, while the right rectangle commutes by functoriality of $- \otimes -$. The two outside compositions are the two ways to chase x around diagram 1, so the diagram commutes as desired. $\square$

4.2. **Commutative monoid objects in $\mathcal{SH}$ and their associated rings.** We have shown that $\pi_*(E)$ is an A-graded ring when (E, μ, e) is a monoid object in $\mathcal{SH}$. A natural question that arises is: In what sense is $\pi_*(E)$ "graded commutative" if (E, μ, e) is a commutative monoid object in $\mathcal{SH}$? It turns out that it satisfies a rather strong commutativity condition. In this subsection, we will show that $\pi_*(E)$ is an *A-graded anticommutative ring*, in the following sense:

Definition 4.5. An *A-graded anticommutative ring* is an A-graded ring R along with an assignment $\theta : A \times A \to R_0^\times$ sending $(a, b) \mapsto \theta_{a,b}$ such that for all $a, b, c \in A$,

- $\theta_{a,0} = \theta_{0,a} = 1$,
- $\theta_{a,b}^{-1} = \theta_{b,a}$,
- $\theta_{a,b} \cdot \theta_{a,c} = \theta_{a,b+c}$ and $\theta_{b,a} \cdot \theta_{c,a} = \theta_{b+c,a}$, and
- for all homogeneous x and y in R,
$$x \cdot y = y \cdot x \cdot \theta_{|x|,|y|}.$$

Given two A-graded anticommutative rings (R, θ) and (R', θ'), an A-graded ring homomorphism $f : R \to R'$ is a homomorphism of A-graded anticommutative rings if it satisfies $f \circ \theta = \theta'$. We write $\mathbf{GrCRing}^A$ for the resulting category.

In fact, the above definition was entirely motivated by the work we will do here. An interesting fact is that the initial object in the category $\mathbf{GrCRing}^A$ is the group algebra $\mathbb{Z}[A \wedge A]$ viewed as an A-graded ring concentrated in degree 0, where here by "$A \wedge A$" we mean the quotient of $A \otimes_{\mathbb{Z}} A$ by the subgroup generated by the elements $a \otimes b + b \otimes a$ for $a, b \in A$. The element $\theta_{a,b} \in \mathbb{Z}[A \wedge A]$ is $a \wedge b = -b \wedge a$, where here $a \wedge b$ denotes the image of the element $a \otimes b$ under the quotient map $A \otimes_{\mathbb{Z}} A \twoheadrightarrow A \wedge A$.

We will show that not only is $\pi_*(E)$ an A-graded anticommutative ring, but it is an A-graded anticommutative algebra over the stable homotopy ring $\pi_*(S)$, defined as follows:

Definition 4.6. Given an A-graded anticommutative ring (R, θ) (Definition 4.5), we write R-$\mathbf{GCA}^A$ to denote the slice category $(R, \theta)/\mathbf{GrCRing}^A$ under (R, θ). Explicitly:

- The objects are pairs (S, φ) called *A-graded anticommutative R-algebras*, where S is an A-graded ring and $\varphi : R \to S$ is an A-graded ring homomorphism such that for all $x \in S_a$ and $y \in S_b$, we have
$$x \cdot y = y \cdot x \cdot \varphi(\theta_{a,b}),$$
- The morphisms $(S, \varphi) \to (S', \varphi')$ are A-graded ring homomorphisms $f : S \to S'$ such that $f \circ \varphi = \varphi'$.

Note that our notation for the category R-$\mathbf{GCA}^A$ is somewhat deficient, as there may be multiple choices of families of units $\theta_{a,b} \in R_0$ satisfying the required properties which give rise to strictly different categories, as the following example illustrates:

Example 4.7. Consider $R = \mathbb{Z}$ as a ring graded over $A = \mathbb{Z}$ concentrated in degree 0, and let $\theta_{n,m} := (-1)^{n \cdot m}$ for all $n, m \in \mathbb{Z}$, then R-$\mathbf{GCA}^A$ is simply the standard category of graded anticommutative rings, i.e., $\mathbb{Z}$-graded rings R such that for all homogeneous $x, y \in R$, $x \cdot y = y \cdot x \cdot (-1)^{|x||y|}$. On the other hand, if we instead define $\theta_{n,m} = 1$ for all $n, m \in \mathbb{Z}$, then the resulting category R-$\mathbf{GCA}^A$ becomes the category of strictly commutative $\mathbb{Z}$-graded rings.

Like the standard category of $\mathbb{Z}$-graded anticommutative rings, it turns out that the category R-$\mathbf{GCA}^A$ has many nice properties. In particular, in Appendix B.4 we show that R-$\mathbf{GCA}^A$ has finite coproducts and pushouts, and as in the standard category of (graded anti)commutative rings, they are formed by taking the underlying tensor product of bimodules and endowing it with a (graded anti)commutative multiplication. The details of this contruction are straightforward but somewhat tedious, so even in the appendix we simply outline what needs to be shown, and leave it to the reader to verify the minute details if they desire.

The rest of this subsection will be devoted to proving that for each commutative monoid object (E, μ, e) in $\mathcal{SH}$, $\pi_*(E)$ is an A-graded anticommutative algebra over the A-graded anticommutative ring $\pi_*(S)$, with structure map $\pi_*(e) : \pi_*(S) \to \pi_*(E)$. Before continuing, we explain how these facts manifest themselves in the classical, motivic, and equivariant stable homotopy categories:

Example 4.8 ([24, Proposition 0.1])**.** In the clasical stable homotopy category, given any commutative ring spectrum (E, μ, e), $\pi_*(E)$ is a $\mathbb{Z}$-graded anticommutative ring in the standard sense, i.e., $\theta_{a,b} = (-e)^{ab} \in \pi_0(E)$ for all $a, b \in \mathbb{Z}$, so that the graded commutativity formula for $\pi_*(E)$ reads

$$x \cdot y = y \cdot x \cdot (-1)^{|x||y|}.$$

Example 4.9 ([10, pg. 3])**.** In the motivic stable homotopy category, there exists an element $\epsilon \in \pi_{0,0}(S)$ and a standard family of $\phi_{a,b}$'s such that that $\pi_{*,*}(S)$ is a $\mathbb{Z}^2$-graded anticommutative ring and the element $\theta_{(a_1,a_2),(b_1,b_2)} \in \pi_{0,0}(S)$ is given by $(-1)^{a_1 b_1}(-\epsilon)^{a_2 b_1 - a_1 b_2 + a_2 b_2}$. In particular, given a motivic ring spectrum (E, μ, e) and homogeneous elements $x \in \pi_{a_1,a_2}(E)$ and $y \in \pi_{b_1,b_2}(E)$, we have

$$x \cdot y = y \cdot x \cdot (-1)^{a_1 b_1} \cdot (e \circ (-\epsilon))^{a_2 b_1 - a_1 b_2 + a_2 b_2}.$$

For motivic ring spectra (E, μ, e) such that $e \circ \epsilon = -e$ (for example, for the motivic mod-p Eilenberg-MacLane spectrum), this formula becomes

$$x \cdot y = y \cdot x \cdot (-1)^{a_1 b_1}.$$

For readers interested in learning more about the different possible graded anticommutativity structures on $\pi_{*,*}(S)$ in the motivic stable homotopy category, we refer the reader to the paper [10] of Dugger, Dundas, Isaksen, and Østvær. There also some of the graded anticommnutativity properties of the C_2-equivariant stable homotopy category are discussed in relation to the motivic stable homotopy category (see Remarks 2 & 3). In general, the graded commutativity properties of the G-equivariant stable homotopy category are highly dependent on the group G, and there is not a standard choice of coherent family of $\phi_{a,b}$'s in this setting. Thankfully, the literature on equivariant stable homotopy theory is usually quite explicit about keeping track of the $\theta_{a,b}$'s (painfully, this is not the case in the motivic setting, where graded commutativity issues are often sidelined).

Now, we continue on with our proof of the graded commutativity properties of $\pi_*(E)$ in $\mathcal{SH}$. To start with, we identify the elements $\theta_{a,b} \in \pi_0(S)$, and show they control anticommutativity in $\pi_*(E)$ for E a commutative monoid object:

Proposition 4.10. *For all $a, b \in A$ there exists an element $\theta_{a,b} \in \pi_0(S) = [S, S]$ such that given any commutative monoid object (E, μ, e) in $\mathcal{SH}$, the A-graded ring structure on $\pi_*(E)$ (Proposition 4.1) has a commutativity formula given by*

$$x \cdot y = y \cdot x \cdot (e \circ \theta_{a,b})$$

for all $x \in \pi_a(E)$ and $y \in \pi_b(E)$.

Proof. Given $a, b \in A$, define $\theta_{a,b} \in \mathrm{Aut}(S)$ to be the composition

$$S \xrightarrow{\cong} S^{-a-b} \otimes S^a \otimes S^b \xrightarrow{S^{-a-b} \otimes \tau} S^{-a-b} \otimes S^b \otimes S^a \xrightarrow{\cong} S,$$

where the outermost maps are the unique maps specified by Remark 2.4. Now let (E, μ, e), x, and y as in the statement of the proposition, and consider the following diagram

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] \arrow[dd, dashed, "\phi_{b,a}^{-1}\circ\tau\circ\phi_{a,b}"'] & S^a \otimes S^b \arrow[r, "x\otimes y"] \arrow[dd, "\tau"'] & E \otimes E \arrow[dd, "\tau"'] \arrow[dr, "\mu"] & \\
& & & E \\
S^{a+b} \arrow[r, "\phi_{b,a}"] & S^b \otimes S^a \arrow[r, "y\otimes x"] & E \otimes E \arrow[ur, "\mu"'] &
\end{tikzcd}$$

The left square commutes by definition. The middle square commutes by naturality of the symmetry isomorphism. Finally, the right square commutes by commutativity of E. Unravelling definitions, we have shown that under the product on $\pi_*(E)$ induced by the $\phi_{a,b}$'s,

$$x \cdot y = (y \cdot x) \circ (\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b}).$$

Thus, in order to show the desired result it further suffices to show that

$$(y \cdot x) \circ (\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b}) = y \cdot x \cdot (e \circ \theta_{a,b}).$$

Consider the following diagram:

$$\begin{tikzcd}
S^{a+b} \arrow[rr, "\phi_{a,b}"] \arrow[d, "\cong"'] & & S^a \otimes S^b \arrow[d, "\tau"] \\
S^b \otimes S^a \otimes S^{-a-b} \otimes S^a \otimes S^b \arrow[urr, "\cong"'] \arrow[d, "S^b\otimes S^a\otimes S^{-a-b}\otimes\tau"'] & & S^b \otimes S^a \arrow[d, "\phi_{b,a}^{-1}"] \arrow[dl, equal] \\
S^b \otimes S^a \otimes S^{-a-b} \otimes S^b \otimes S^a \arrow[r, "\cong"] \arrow[urr, "\cong"] & S^b \otimes S^a \arrow[dr, "y\otimes x"] \arrow[ddl, "y\otimes x\otimes e"'] & S^{a+b} \arrow[l, "\phi_{b,a}"'] \\
& & E \otimes E \arrow[dll, "E\otimes E\otimes e"'] \arrow[d, equal] \\
E \otimes E \otimes E \arrow[rr, "E\otimes\mu"] \arrow[d, "\mu\otimes E"'] & & E \otimes E \arrow[d, "\mu"] \\
E \otimes E \arrow[rr, "\mu"'] & & E
\end{tikzcd}$$

Here any map simply labelled $\cong$ is an appropriate composition of copies of $\phi_{a,b}$'s, associators, and their inverses, so that each of these maps are necessarily unique by Remark 2.4. The triangles in the top large rectangle commutes by coherence for the $\phi_{a,b}$'s. The parallelogram commutes by naturality of τ and coherence of the of $\phi_{a,b}$'s. The middle skewed triangle commutes by functoriality of $- \otimes -$. The triangle below that commutes by unitality of μ. Finally, the bottom rectangle commmutes by associativity of μ. Hence, by unravelling definitions and applying functoriality of $- \otimes -$, we get that the right composition is $(y \cdot x) \circ (\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b})$, while the left composition is $y \cdot x \cdot (e \circ \theta_{a,b})$, so they are equal as desired. $\square$

Now, it remains to show that the assignment $\theta : A^2 \to \pi_0(S)$ descends/restricts to a group homomorphism $A \wedge A \to \pi_0(S)^\times$, i.e., that it satisfies the first three conditions outlined in Definition 4.5. First, we prove the following useful lemma:

Lemma 4.11. *Suppose we have homogeneous elements $x, y \in \pi_*(S)$ with x of degree 0 (so x is a map $S \to S$ and y is a map $S^a \to S$ for some $a \in A$), then we have $x \cdot y = y \cdot x = x \circ y$ (where the $\cdot$ denotes the product given in Proposition 4.1).*

Proof. As morphisms, y is an arrow $S^a \to S$ for some a in A, and x is a morphism $S \to S$. Then consider the following diagram:

$$\begin{tikzcd}
S \otimes S^a \arrow[dr, "S\otimes y"] \arrow[dd, "y\otimes x"'] & & S^a \arrow[ll, "\phi_{0,a}=\lambda_{S^a}^{-1}"'] \arrow[rr, "\phi_{a,0}=\rho_{S^a}^{-1}"] \arrow[d, "y"'] & & S^a \otimes S \arrow[dl, "y\otimes S"'] \arrow[dd, "x\otimes y"] \\
& S \otimes S \arrow[r, "\lambda_S=\rho_S"] \arrow[dl, "x\otimes S"'] & S \arrow[d, "x"'] & S \otimes S \arrow[l, "\rho_S=\lambda_S"'] \arrow[dr, "S\otimes x"] & \\
S \otimes S \arrow[rr, "\phi_{0,0}^{-1}=\rho_S"'] & & S & & S \otimes S \arrow[ll, "\phi_{0,0}^{-1}=\lambda_S"]
\end{tikzcd}$$

The trapezoids commute by naturality of the unitors, and the triangles commute by functoriality of $-\otimes-$. The outside compositions are $y \cdot x$ on the left and $x \cdot y$ on the right, and the middle composition is $x \circ y$, so indeed we have $y \cdot x = x \cdot y = x \circ y$, as desired. □

Now, we will check the rest of the conditions in Definition 4.5 1-by-1.

Lemma 4.12. *Given $a \in A$, we have $\theta_{0,a} = \theta_{a,0} = \mathrm{id}_S$.*

Proof. Recall $\theta_{a,0}$ is the composition

$$S \xrightarrow{\phi_{-a,a}} S^{-a} \otimes S^a \xrightarrow{S^{-a}\otimes\phi_{a,0}} S^{-a} \otimes (S^a \otimes S) \xrightarrow{S^{-a}\otimes\tau} S^{-a} \otimes (S \otimes S^a) \xrightarrow{S^{-a}\otimes\phi_{0,a}^{-1}} S^{-a} \otimes S^a \xrightarrow{\phi_{-a,a}^{-1}} S$$

By the coherence theorem for symmetric monoidal categories and the fact that $\phi_{a,0}$ and $\phi_{0,a}$ coincide with the unitors, we have that the composition

$$S^a \xrightarrow{\phi_{a,0}=\rho_{S^a}^{-1}} S^a \otimes S \xrightarrow{\tau} S \otimes S^a \xrightarrow{\phi_{0,a}^{-1}=\lambda_{S^a}} S^a$$

is precisely the identity map, so by functoriality of $-\otimes-$, we have that $\theta_{a,0}$ is the composition

$$S \xrightarrow{\phi_{-a,a}} S^{-a} \otimes S^a \xrightarrow{=} S^{-a} \otimes S^a \xrightarrow{\phi_{-a,a}^{-1}} S.$$

Hence $\theta_{a,0} = \mathrm{id}_S$, as desired. An entirely analagous argument yields that $\theta_{0,a} = \mathrm{id}_S$. □

Lemma 4.13. *Let $a, b \in A$. Then $\theta_{a,b} \cdot \theta_{b,a} = \mathrm{id}_S$.*

Proof. By Lemma 4.11, it suffices to show that $\theta_{a,b} \circ \theta_{b,a} = \mathrm{id}_S$. To see this, consider the following diagram:

$$\begin{tikzcd}
S \arrow[r, "\phi"] \arrow[dddrrr, equal] & S^{-a-b} \otimes S^b \otimes S^a \arrow[r, "S^{-a-b}\otimes\tau"] \arrow[ddrr, equal] & S^{-a-b} \otimes S^a \otimes S^b \arrow[r, "\phi"] \arrow[dr, equal] & S \arrow[d, "\phi"] \\
& & & S^{-a-b} \otimes S^a \otimes S^b \arrow[d, "S^{-a-b}\otimes\tau"] \\
& & & S^{-a-b} \otimes S^b \otimes S^a \arrow[d, "\phi"] \\
& & & S
\end{tikzcd}$$

Here we are suppressing associators, and any map labelled ϕ is the appropriate composition of $\phi_{a,b}$'s, unitors, associators, identities, and their inverses (see Remark 2.4). Clearly each region commutes, the middle by the fact that $\tau^2 = \mathrm{id}$, and the other two regions by coherence for the ϕ's. Thus we have shown $\theta_{a,b} \cdot \theta_{b,a} = \theta_{a,b} \cdot \theta_{b,a} = \mathrm{id}_S$, as desired. □

Lemma 4.14. *Let $a, b, c \in A$. Then $\theta_{a,b} \cdot \theta_{a,c} = \theta_{a,b+c}$ and $\theta_{b,a} \cdot \theta_{c,a} = \theta_{b+c,a}$.*

Proof. First we show $\theta_{a,b}\cdot\theta_{a,c} = \theta_{a,b+c}$. By Lemma 4.11, it suffices to show that $\theta_{a,b}\circ\theta_{a,c} = \theta_{a,b+c}$. To see this, consider the following diagram:
(2)

$$\begin{tikzcd}
S \arrow[rr, "\phi"] \arrow[dd, "\phi"'] & & S^{-a-c}S^aS^c \arrow[rr, "S^{-a-c}\tau"] \arrow[dd, "\phi"'] & & S^{-a-c}S^cS^a \arrow[rr, "\phi"] \arrow[dd, "\phi"] & & S \arrow[dd, "\phi"] \\
& (A) & & (B) & & (C) & \\
S^{-a-b-c}S^aS^{b+c} \arrow[rr, "\phi"] \arrow[dd, "S^{-a-b-c}\tau"'] & & S^{-a-c}S^{-b}S^aS^bS^c \arrow[rr, "S^{-a-c}\tau_{S^{-b}S^aS^b,S^c}"] \arrow[dd, "S^{-a-c}S^{-b}\tau_{S^a,S^bS^c}"'] & & S^{-a-c}S^cS^{-b}S^aS^b \arrow[dd, "S^{-a-c}S^cS^{-b}\tau"] & & S^{-a-b}S^aS^b \arrow[ll, "\phi"'] \arrow[dd, "S^{-a-b}\tau"] \\
& (D) & & (E) & & (F) & \\
S^{-a-b-c}S^{b+c}S^a \arrow[rr, "\phi"] \arrow[dddd, "\phi"'] & & S^{-a-c}S^{-b}S^bS^cS^a \arrow[rr, "S^{-a-c}\tau_{S^{-b}S^b,S^c}S^a"] & & S^{-a-c}S^cS^{-b}S^bS^a & & S^{-a-b}S^bS^a \arrow[ll, "\phi"'] \arrow[dddd, "\phi"] \\
& & & (G) & & & \\
& & S^{-a-c}S^cS^a \arrow[uu, "\phi"] \arrow[rr, equal] & & S^{-a-c}S^cS^a \arrow[uu, "\phi"'] & & \\
& & & (H) & & & \\
S \arrow[rrrrrr, equal] & & & & & & S
\end{tikzcd}$$

Here we are omitting $\otimes$ from the notation (so that the diagram fits on the page), and each occurrence of an arrow labelled ϕ indicates it is the unique arrow that can be obtained as a formal composition of tensor products of copies of $\phi_{a,b}$'s, unitors, associators, and their inverses (Remark 2.4). Clearly the composition going around the top and then the right is $\theta_{a,b}\circ\theta_{a,c}$ while the composition going left around the bottom is $\theta_{a,b+c}$. Thus, we wish to show the above diagram commutes.

Regions (A), (C), and (H) commute by coherence for the ϕ's (see previous remark). Region (E) commutes by coherence for the τ's. To see region (B) commutes, consider the following diagram, which commutes by naturality of τ:

$$\begin{tikzcd}
S^{-a-c}S^aS^c \arrow[r, "S^{-a-c}\tau"] \arrow[d, "S^{-a-c}\phi_{a-b,b}S^c"'] & S^{-a-c}S^cS^a \arrow[d, "S^{-a-c}S^c\phi_{a-b,b}"] \\
S^{-a-c}S^{a-b}S^bS^c \arrow[r, "S^{-a-c}\tau_{S^{a-b}S^b,S^c}"] \arrow[d, "S^{-a-c}\phi_{-b,a}S^bS^c"'] & S^{-a-c}S^cS^{a-b}S^b \arrow[d, "S^{-a-c}S^c\phi_{-b,a}S^b"] \\
S^{-a-c}S^{-b}S^aS^bS^c \arrow[r, "S^{-a-c}\tau_{S^{-b}S^aS^b,S^c}"] & S^{-a-c}S^cS^{-b}S^aS^b
\end{tikzcd}$$

To see region (D) commutes, note that it is simply the square

$$\begin{tikzcd}
S^{-a-b-c}S^aS^{b+c} \arrow[r, "\phi_{-a-c,-b}S^a\phi_{b,c}"] \arrow[d, "S^{-a-b-c}\tau"'] & S^{-a-c}S^{-b}S^aS^bS^c \arrow[d, "S^{-a-c}S^{-b}\tau_{S^a,S^bS^c}"] \\
S^{-a-b-c}S^{b+c}S^a \arrow[r, "\phi_{-a-c,-b}\phi_{b,c}S^a"] & S^{-a-c}S^{-b}S^bS^cS^a
\end{tikzcd}$$

This diagram commutes by naturality of τ. To see region (F) commutes, consider the following diagram, which commutes by functoriality of $-\otimes-$:

$$\begin{tikzcd}
S^{-a-c}S^cS^{-b}S^aS^b \arrow[d, "S^{-a-c}S^cS^{-b}\tau"'] & S^{-a-c}S^{c-b}S^aS^b \arrow[l, "S^{-a-c}\phi_{c,-b}S^aS^b"'] \arrow[d, "S^{-a-c}S^{c-b}\tau"] & S^{-a-b}S^aS^b \arrow[l, "\phi_{-a-c,c-b}S^aS^b"'] \arrow[d, "S^{-a-b}\tau"] \\
S^{-a-c}S^cS^{-b}S^bS^a & S^{-a-c}S^{c-b}S^bS^a \arrow[l, "S^{-a-c}\phi_{c,-b}S^bS^a"'] & S^{-a-b}S^bS^a \arrow[l, "\phi_{-a-c,c-b}S^bS^a"']
\end{tikzcd}$$

Finally, to see region (G) commutes, consider the following diagram:

$$\begin{tikzcd}
S^{-a-c}S^{-b}S^bS^cS^a \arrow[r, "S^{-a-c}\tau_{S^{-b}S^b,S^c}S^a"] & S^{-a-c}S^cS^{-b}S^bS^a \\
S^{-a-c}SS^cS^a \arrow[u, "S^{-a-c}\phi_{-b,b}S^cS^a"] \arrow[r, "S^{-a-c}\tau_{S,S^c}S^a"] & S^{-a-c}S^cSS^a \arrow[u, "S^{-a-c}S^c\phi_{-b,b}S^a"'] \\
S^{-a-c}S^cS^a \arrow[u, "S^{-a-c}\phi_{0,c}S^a=S^{-a-c}\lambda_{S^c}^{-1}S^a"] \arrow[r, equal] & S^{-a-c}S^cS^a \arrow[u, "S^{-a-c}\phi_{c,0}S^a=S^{-a-c}S\rho_{S^c}^{-1}S^a"']
\end{tikzcd}$$

The top region commutes by naturality of τ, while the bottom region commutes by coherence for a symmetric monoidal category. Thus, we have shown that diagram (2) commutes, so that $\theta_{a,b}\cdot\theta_{a,c} = \theta_{a,b+c}$, as desired. Now, to see that $\theta_{b,a}\cdot\theta_{c,a} = \theta_{b+c,a}$, note that

$$\theta_{b,a}\cdot\theta_{c,a} \overset{(*)}{=} \theta_{a,b}^{-1}\cdot\theta_{a,c}^{-1} = (\theta_{a,c}\cdot\theta_{a,b})^{-1} = \theta_{a,b+c}^{-1} \overset{(*)}{=} \theta_{b+c,a},$$

where each occurrence of $(*)$ is Lemma 4.13. □

To recap, we have shown that the assignment $\theta : A^2 \to \pi_0(S)^\times$ satisfies the following for all $a, b, c \in A$:

- $\theta_{a,0} = \theta_{0,a} = 1$,
- $\theta_{a,b}^{-1} = \theta_{b,a}$,
- $\theta_{a,b}\cdot\theta_{a,c} = \theta_{a,b+c}$ and $\theta_{b,a}\cdot\theta_{c,a} = \theta_{b+c,a}$, and
- for all homogeneous x and y in $\pi_*(S)$,

$$x\cdot y = y\cdot x\cdot\theta_{|x|,|y|}.$$

Thus, the stable homotopy ring $\pi_*(S)$ is an A-graded anticommutative ring, as desired. Now, we just have a few details left to check in order to conclude that $\pi_*(E)$ is an A-graded anticommutative $\pi_*(S)$-algebra for E a commutative monoid object in $\mathcal{SH}$:

Proposition 4.15. *The assignment $(E,\mu,e)\mapsto(\pi_*(E),\pi_*(e))$ yields a functor*

$$\pi_* : \mathbf{CMon}_{\mathcal{SH}} \to \pi_*(S)\text{-}\mathbf{GCA}^A$$

from the category of commutative monoid objects in $\mathcal{SH}$ (Definition C.3) to the category of A-graded anticommutative $\pi_*(S)$-algebras (Definition 4.6).

Proof. By Proposition 4.1, we know that π_* yields a functor from $\mathbf{CMon}_{\mathcal{SH}}$ to A-graded rings. Furthermore, by Proposition 4.10, we know that for all homogeneous $x, y \in \pi_*(E)$ that

$$x\cdot y = y\cdot x\cdot(e\circ\theta_{|x|,|y|}) = y\cdot x\cdot\pi_*(e)(\theta_{|x|,|y|}),$$

as desired. Thus, it remains to show that $\pi_*(e) : \pi_*(S)\to\pi_*(E)$ is an A-graded ring homomorphism for any (commutative) monoid object (E,μ,e) in $\mathcal{SH}$, and that given a monoid homomorphism $f : (E_1,\mu_1,e_1)\to(E_2,\mu_2,e_2)$ in $\mathbf{CMon}_{\mathcal{SH}}$, that $\pi_*(f)$ satisfies $\pi_*(f)\circ\pi_*(e_1) = \pi_*(e_2)$. The latter clearly holds, as since f is a monoid homomorphism, we have $f\circ e_1 = e_2$, so that

$$\pi_*(f)\circ\pi_*(e_1) = \pi_*(f\circ e_1) = \pi_*(e_2).$$

Furthermore, since $e : S \to E$ is a monoid object homomorphism (Lemma C.6), we know that $\pi_*(e) : \pi_*(S) \to \pi_*(E)$ is an A-graded ring homomorphism by Proposition 4.1. □

5. Some important theorems in $\mathcal{SH}$

So far, we have already identified a good amount of structure on the objects $\pi_*(E)$, $E_*(E) = \pi_*(E \otimes E)$, and $E_*(X)$ for E a (commutative) monoid object and X an object in $\mathcal{SH}$. Namely, we have shown that $\pi_*(E)$ and $E_*(E) = \pi_*(E \otimes E)$ are canonically A-graded anticommutative algebras over the stable homotopy ring (Proposition 4.15), and that $E_*(X)$ is canonically an A-graded left $\pi_*(E)$-module (Proposition 4.3). We would like to identify even more structure on these objects, namely, in Section 6, we will show that the pair $(E_*(E), \pi_*(E))$ is an *A-graded anticommutative Hopf algebroid*, over which $E_*(X)$ is an A-graded left comodule. To that end, we need two important theorems, namely, we need analogs of the Künneth isomorphism and the universal coefficient theorem from algebraic topology. This section is dedicated to formulating and proving these theorems. The proofs of these theorems are arguably the most technical and difficult in this paper, so we will be especially careful to give them in their full and explicit detail.

5.1. **A Künneth isomorphism.** The goal of this subsection will be to prove the following theorem, which, given a monoid object (E, μ, e) and objects Z and W in $\mathcal{SH}$, relates the Z-homology of E and the E-homology of W to $\pi_*(Z \otimes E \otimes W)$:

Theorem 5.1 (The Künneth isomorphism)**.** *Let (E, μ, e) be a monoid object and Z and W objects in $\mathcal{SH}$. Then if*

- *$Z_*(E)$ is a flat right $\pi_*(E)$-module (via Proposition 4.3) and W is cellular (Definition 3.1),* or
- *$E_*(W)$ is a flat left $\pi_*(E)$-module (via Proposition 4.3) and Z is cellular,*

then there is a natural A-graded isomorphism of A-graded abelian groups, called the Künneth isomorphism*:*

$$\Phi_{Z,W} : Z_*(E) \otimes_{\pi_*(E)} E_*(W) \xrightarrow{\cong} \pi_*(Z \otimes E \otimes W).$$

There is much work to be done. First, we construct the map and show it is natural:

Proposition 5.2. *Let (E, μ, e) be a monoid object and Z and W be objects in $\mathcal{SH}$. Then there is an A-graded homomorphism of abelian groups*

$$\Phi_{Z,W} : Z_*(E) \otimes_{\pi_*(E)} E_*(W) \to \pi_*(Z \otimes E \otimes W)$$

which given homogeneous elements $x : S^a \to Z \otimes E$ in $Z_*(E) = \pi_*(Z \otimes E)$ and $y : S^b \to E \otimes W$ in $E_*(W) = \pi_*(E \otimes W)$, sends the homogeneous pure tensor $x \otimes y$ in $Z_*(E) \otimes_{\pi_*(E)} E_*(W)$ to the composition

$$\Phi_{Z,W}(x \otimes y) : S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x \otimes y} Z \otimes E \otimes E \otimes W \xrightarrow{Z \otimes \mu \otimes W} Z \otimes E \otimes W$$

Furthermore, this homomorphism is natural in both Z and W.

Proof. By Lemma B.14, in order to get an A-graded homomorphism

$$\Phi_{Z,W} : Z_*(E) \otimes_{\pi_*(E)} E_*(W) \to \pi_*(Z \otimes E \otimes W),$$

it suffices to define an assignment $P : Z_*(E) \times E_*(W) \to \pi_*(Z \otimes E \otimes W)$ on homogeneous elements (which we have), and show that it is additive in each argument for homogeneous elements of the same degree, and that for all homogeneous $z \in Z_*(E)$, $r \in \pi_*(E)$, and $w \in E_*(W)$ that $P(zr, w) = P(z, rw)$, where concatenation denotes the module action.

First, note that by Lemma 4.2 it is straightforward to see that the assignment commutes with addition of maps of the same degree in each argument. Now, let $a, b, c \in A$, $z : S^a \to Z \otimes E$,

$w : S^b \to E \otimes W$, and $r : S^c \to E$. Then we wish to show $P(zr, w) = P(z, rw)$. Consider the following diagram (where here we are passing to a symmetric strict monoidal category):

$$\begin{tikzcd}
 & & & Z \otimes E \otimes E \otimes W \arrow[d, "Z\otimes\mu\otimes W"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^a \otimes S^c \otimes S^b \arrow[r, "z\otimes r\otimes w"] & Z \otimes E \otimes E \otimes E \otimes W \arrow[ur, "Z\otimes\mu\otimes E\otimes W"] \arrow[dr, "Z\otimes E\otimes\mu\otimes W"'] & Z \otimes E \otimes W \\
 & & & Z \otimes E \otimes E \otimes W \arrow[u, "Z\otimes\mu\otimes W"']
\end{tikzcd}$$

It commutes by associativity of μ. By functoriality of $- \otimes -$, the top composition is given by $P(zr, w)$ and the bottom composition is $P(z, rw)$, so they are equal as desired. Thus, by Lemma B.14 we get the desired A-graded homomorphism $\pi_*(Z \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W) \to \pi_*(Z \otimes E \otimes W)$.

Next, we would like to show that this homomorphism is natural in Z. Let $f : Z \to Z'$ in $\mathcal{SH}$. Then we would like to show the following diagram commutes:

$$\begin{tikzcd}
\pi_*(Z \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W) \arrow[r, "\Phi_{Z,W}"] \arrow[d, "\pi_*(f\otimes E)\otimes\pi_*(E\otimes W)"'] & \pi_*(Z \otimes E \otimes W) \arrow[d, "\pi_*(f\otimes E\otimes W)"] \\
\pi_*(Z' \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W) \arrow[r, "\Phi_{Z',W}"] & \pi_*(Z' \otimes E \otimes W)
\end{tikzcd} \tag{3}$$

As all the maps here are homomorphisms, in order to show it commutes, it suffices to chase generators around the diagram. In particular, suppose we are given $z : S^a \to Z \otimes E$ and $w : S^b \to E \otimes W$, and consider the following diagram exhibiting the two possible ways to chase $z \otimes w$ around the diagram (as usual, we are passing to a symmetric strict monoidal category):

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] & S^a \otimes S^b \arrow[r, "z\otimes w"] & Z \otimes E \otimes E \otimes W \arrow[r, "Z\otimes\mu\otimes W"] \arrow[d, "f\otimes E\otimes E\otimes W"'] & Z \otimes E \otimes W \arrow[d, "f\otimes E\otimes W"] \\
 & & Z \otimes E \otimes E \otimes W \arrow[r, "Z\otimes\mu\otimes W"] & Z' \otimes E \otimes W
\end{tikzcd}$$

This diagram commutes by functoriality of $- \otimes -$. Thus we have that diagram (3) does indeed commute, so that $\Phi_{Z,W}$ is natural in Z as desired. Showing that $\Phi_{Z,W}$ is natural in W is entirely analagous. $\square$

Now, before proving the Künneth map is an isomorphism under the conditions given in Theorem 5.1, we prove the following lemmas:

Lemma 5.3. *Let (E, μ, e) be a monoid object and Z and W be objects in $\mathcal{SH}$. Then for all $a \in A$, the following diagram commutes*

$$\begin{tikzcd}
Z_*(E) \otimes_{\pi_*(E)} E_*(\Sigma^a W) \arrow[r, "Z_*(E)\otimes_{\pi_*(E)} t_a^W"] \arrow[d, "\Phi_{Z,\Sigma^a W}"'] & Z_*(E) \otimes_{\pi_*(E)} E_{*-a}(W) \arrow[d, "\Phi_{Z,W}"] \\
\pi_*(Z \otimes E \otimes \Sigma^a W) \arrow[d, equal] & \pi_{*-a}(Z \otimes E \otimes W) \arrow[d, equal] \\
(Z \otimes E)_*(\Sigma^a W) \arrow[r, "t_a^W"] & (Z \otimes E)_{*-a}(W)
\end{tikzcd}$$

where the maps t_a are constructed and proven to be A-graded isomorphisms of abelian groups in Lemma 4.4.

Proof. Note that in Lemma 4.4, it is shown that $t_a^W : E_*(\Sigma^a W) \to E_{*-a}(W)$ is not just an A-graded isomorphism of abelian groups, but it is furthermore a left $\pi_*(E)$-module isomorphism. Thus, the top arrow in the above diagram is well-defined. Since all the arrows involved are A-graded homomorphisms, in order to show the diagram commutes it suffices to chase a pure homogeneous tensor around, as they generate the top left object. To that end, let $x : S^b \to Z \otimes E$ in $Z_*(E)$ and $y : S^c \to E \otimes S^a \otimes W$ in $E_*(\Sigma^a W)$, and consider the following diagram exhibiting the two ways to chase $x \otimes y$ around:

$$
\begin{tikzcd}
S^{b+c-a} \arrow[d, "\phi"] & Z \otimes E \otimes E \otimes W \otimes S^a \otimes S^{-a} \arrow[r, "Z\otimes E\otimes E\otimes W\otimes\phi_{a,-a}^{-1}"] \arrow[ddr, "Z\otimes\mu\otimes W\otimes S^a\otimes S^{-a}"] & Z \otimes E \otimes E \otimes W \arrow[d, "Z\otimes\mu\otimes W"] \\
S^b \otimes S^c \otimes S^{-a} \arrow[r, "x\otimes y\otimes S^{-a}"] & Z \otimes E \otimes E \otimes S^a \otimes W \otimes S^{-a} \arrow[u, "Z\otimes E\otimes E\otimes\tau\otimes S^{-a}"'] \arrow[d, "Z\otimes\mu\otimes S^a\otimes W\otimes S^{-a}"'] & Z \otimes E \otimes W \\
& Z \otimes E \otimes S^a \otimes W \otimes S^{-a} \arrow[r, "Z\otimes E\otimes\tau\otimes S^{-a}"'] & Z \otimes E \otimes W \otimes S^a \otimes S^{-a} \arrow[u, "Z\otimes E\otimes W\otimes\phi_{a,-a}^{-1}"']
\end{tikzcd}
$$

Each triangle commutes by functoriality of $- \otimes -$, so the diagram commutes as desired. $\square$

Lemma 5.4. *Given a monoid object (E, μ, e) and an object X in $\mathcal{SH}$, for all $a \in A$ the A-graded isomorphisms*

$$s^a_{X\otimes E} : \pi_*(\Sigma^a X \otimes E) \to \pi_{*-a}(X \otimes E)$$

from Definition 2.9 are isomorphisms of right $\pi_*(E)$-modules, where here $\pi_*(\Sigma^a X \otimes E)$ and $\pi_*(X \otimes E) = X_*(E)$ are considered with their canonical right $\pi_*(E)$-module structure given in Proposition 4.3.

Proof. By additivity, in order to show $s^a_{X\otimes E}$ is a homomorphism of right $\pi_*(E)$-modules, it suffices to show that for all homogeneous $x : S^b \to S^a \otimes X \otimes E$ in $\pi_*(\Sigma^a X \otimes E)$ and $r : S^b \to E$ in $\pi_*(E)$ that $s^a_{X\otimes E}(x \cdot r) = s^a_{X\otimes E}(x) \cdot r$. To that end, consider the following diagram:

$$
\begin{tikzcd}
S^{b+c-a} \arrow[r, "\phi"] & S^{-a} \otimes S^b \otimes S^c \arrow[r, "S^{-a}\otimes x\otimes r"] & S^{-a} \otimes S^a \otimes X \otimes E \otimes E \arrow[r, "S^{-a}\otimes S^a\otimes X\otimes\mu"] \arrow[d, "\phi^{-1}_{-a,a}\otimes X\otimes E\otimes E"'] & S^{-a} \otimes S^a \otimes X \otimes E \arrow[d, "\phi^{-1}_{-a,a}\otimes X\otimes E"] \\
& & X \otimes E \otimes E \arrow[r, "X\otimes\mu"] & X \otimes E
\end{tikzcd}
$$

The top composition is $s^a_{X\otimes E}(x \cdot r)$, while the bottom composition is $s^a_{X\otimes E}(x) \cdot r$. The diagram commutes by functoriality of $- \otimes -$, so that $s^a_{X\otimes E}(x \cdot r) = s^a_{X\otimes E}(x) \cdot r$ as desired, meaning $s^a_{X\otimes E}$ is indeed a right $\pi_*(E)$-module homomorphism. $\square$

Lemma 5.5. *Let (E, μ, e) be a monoid object and Z and W objects in $\mathcal{SH}$, and suppose the Künneth map $\Phi_{Z,W}$ is an isomorphism. Then $\Phi_{\Sigma^a Z,W}$ and $\Phi_{Z,\Sigma^a W}$ are isomorphisms for all $a \in A$, and so are $\Phi_{\Sigma Z,W}$ an $\Phi_{Z,\Sigma W}$.*

Proof. If $\Phi_{Z,W}$ is an isomorphism, it follows that $\Phi_{Z,\Sigma^a W}$ is an isomorphism by Lemma 5.3. On the other hand, in order to see $\Phi_{\Sigma^a Z,W}$ is an isomorphism, consider the following diagram:

$$
\begin{tikzcd}
\pi_*(\Sigma^a Z \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W) \arrow[r, "\Phi_{\Sigma^a Z,W}"] \arrow[d, "s^a_{Z\otimes E}\otimes_{\pi_*(E)}\pi_*(E\otimes W)"'] & \pi_*(\Sigma^a Z \otimes E \otimes W) \arrow[d, "s^a_{Z\otimes E\otimes W}"] \\
\pi_{*-a}(Z \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W) \arrow[r, "\Phi_{Z,W}"] & \pi_{*-a}(Z \otimes E \otimes W)
\end{tikzcd} \tag{4}
$$

Here the vertical arrows are induced via the isomorphisms constructed in Definition 2.9, and the left vertical arrow is well-defined since $s^a_{Z\otimes E}$ is a right $\pi_*(E)$-module homomorphism by Lemma 5.4. Since every arrow in diagram (4) is an isomorphism of abelian groups except the top arrow, in order to show $\Phi_{\Sigma^a Z,W}$ is an isomorphism, it suffices to show the diagram commutes. To that end, since all the arrows are homomorphisms, it suffices to chase a pure homogeneous tensor around. So let $x : S^b \to \Sigma^a Z \otimes E$ and $y : S^c \to E \otimes W$, and consider the following diagram whose outside compositions exhibit the two ways to chase the pure tensor $x \otimes y$ around diagrama (4):

$$\begin{tikzcd}
S^{b+c-a} \arrow[r, "\phi"] & S^{-a} \otimes S^b \otimes S^c \arrow[r, "S^{-a}\otimes x\otimes y"] & S^{-a} \otimes S^a \otimes Z \otimes E \otimes E \otimes W \arrow[r, "S^{-a}\otimes S^a\otimes Z\otimes\mu\otimes W"] \arrow[d, "\phi^{-1}_{-a,a}\otimes Z\otimes E\otimes E\otimes W"'] & S^{-a} \otimes S^a \otimes Z \otimes E \otimes W \arrow[d, "\phi^{-1}_{-a,a}\otimes Z\otimes E\otimes W"] \\
 & & Z \otimes E \otimes E \otimes W \arrow[r, "Z\otimes\mu\otimes W"] & Z \otimes E \otimes W
\end{tikzcd}$$

The diagram commutes by functoriality of $-\otimes-$, so that diagram (4) commutes, meaning $\Phi_{\Sigma^a Z,W}$ is an isomorphism as desired.

Now, it remains to show that $\Phi_{Z,\Sigma W}$ and $\Phi_{\Sigma Z,W}$ are isomorphisms. To that end, consider the following diagram:

$$\begin{tikzcd}
\pi_*(Z \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes \Sigma W) \arrow[r, "\Phi_{Z,\Sigma W}"] \arrow[d, "\pi_*(Z\otimes E)\otimes_{\pi_*(E)}\pi_*(E\otimes\nu_W)"'] & \pi_*(Z \otimes E \otimes \Sigma W) \arrow[d, "\pi_*(Z\otimes E\otimes\nu_W)"] \\
\pi_*(Z \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes \Sigma^{\mathbf{1}} W) \arrow[r, "\Phi_{Z,\Sigma^{\mathbf{1}} W}"'] & \pi_*(Z \otimes E \otimes \Sigma^{\mathbf{1}} W)
\end{tikzcd}$$

It commutes by naturality of Φ. Furthermore, assuming $\Phi_{Z,W}$ is an isomorphism, by what we have shown above we know that $\Phi_{Z,\Sigma^{\mathbf{1}}W}$ is an isomorphism, and since ν_W is an isomorphism, it follows that the above diagram commutes and all arrows except $\Phi_{Z,\Sigma W}$ are isomorphisms, so that $\Phi_{Z,\Sigma W}$ must be an isomorphism itself. Finally, an entirely analagous argument using naturality of Φ with respect to ν_Z yields that $\Phi_{\Sigma Z,W}$ is an isomorphism as well. $\square$

Now, we can finally prove the desired theorem:

Proposition 5.6. *Let (E, μ, e) be a monoid object and Z and W objects in $\mathcal{SH}$. Then if either:*

(1) $Z_*(E)$ is a flat right $\pi_*(E)$-module (via Proposition 4.3) and W is cellular (Definition 3.1), or

(2) $E_*(W)$ is a flat left $\pi_*(E)$-module (via Proposition 4.3) and Z is cellular,

then the natural homomorphism

$$\Phi_{Z,W} : Z_*(E) \otimes_{\pi_*(E)} E_*(W) \to \pi_*(Z \otimes E \otimes W)$$

given in Proposition 5.2 is an isomorphism of abelian groups.

Proof. In this proof, we will freely employ the coherence theorem for symmetric monoidal categories, and we will assume that associativity and unitality of $-\otimes-$ holds up to strict equality. First we will consider the case that $\pi_*(Z \otimes E) = Z_*(E)$ is a flat right $\pi_*(E)$-module and W is cellular. To start, let $\mathcal{E}$ be the collection of objects W in $\mathcal{SH}$ for which $\Phi_{Z,W}$ is an isomorphism. Then in order to show $\mathcal{E}$ contains every cellular object, it suffices to show that $\mathcal{E}$ satisfies the three conditions given for the class of cellular objects in Definition 3.1. First, we need to show that $\Phi_{Z,W}$ is an isomorphism when $W = S^a$ for some $a \in A$. Indeed, consider the A-graded homomorphism

$$\Psi : \pi_*(Z \otimes E \otimes S^a) \to \pi_*(Z \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes S^a)$$

which sends a class $x : S^b \to Z \otimes E \otimes S^a$ in $\pi_b(Z \otimes E \otimes S^a)$ to the pure tensor $\widetilde{x} \otimes \widetilde{e}$, where $\widetilde{x} \in \pi_{b-a}(Z \otimes E)$ is the composition

$$S^{b-a} \xrightarrow{\phi_{b,-a}} S^b \otimes S^{-a} \xrightarrow{x \otimes S^{-a}} Z \otimes E \otimes S^a \otimes S^{-a} \xrightarrow{Z \otimes E \otimes \phi^{-1}_{a,-a}} Z \otimes E$$

and $\widetilde{e} \in \pi_a(E \otimes S^a)$ is the composition

$$S^a \xrightarrow{e \otimes S^a} E \otimes S^a.$$

In order to see Ψ is an (A-graded) homomorphism of abelian groups: Given $x, x' \in \pi_b(Z \otimes E \otimes S^a)$, we would like to show that $\widetilde{x} \otimes \widetilde{e} + \widetilde{x}' \otimes \widetilde{e} = \widetilde{x + x'} \otimes \widetilde{e}$. It suffices to show that $\widetilde{x} + \widetilde{x}' = \widetilde{x + x'}$. To see this, consider the following diagram (again, we are passing to a symmetric strict monoidal category):

$$\begin{tikzcd}
S^{b-a} \arrow[r, "\Delta"] \arrow[d, "\phi_{b-a}"'] & S^{b-a} \oplus S^{b-a} \arrow[d, "\phi_{b,-a} \oplus \phi_{b,-a}"] \\
S^b \otimes S^{-a} \arrow[r, "\Delta"] \arrow[d, "\Delta \otimes S^{-a}"'] & (S^b \otimes S^{-a}) \oplus (S^b \otimes S^{-a}) \arrow[d, "(x \otimes S^{-a}) \oplus (x' \otimes S^{-a})"] \\
(S^b \oplus S^b) \otimes S^{-a} \arrow[ur, "\cong"] \arrow[d, "(x \oplus x') \otimes S^{-a}"'] & (Z \otimes E \otimes S^a \otimes S^{-a}) \oplus (Z \otimes E \otimes S^a \otimes S^{-a}) \arrow[d, "(Z \otimes E \otimes \phi^{-1}_{a,-a}) \oplus (Z \otimes E \otimes \phi^{-1}_{a,-a})"] \arrow[ddl, "\nabla"] \\
((Z \otimes E \otimes S^a) \oplus (Z \otimes E \otimes S^a)) \otimes S^{-a} \arrow[ur, "\cong"] \arrow[d, "\nabla \otimes S^{-a}"'] & (Z \otimes E) \oplus (Z \otimes E) \arrow[d, "\nabla"] \\
Z \otimes E \otimes S^a \otimes S^{-a} \arrow[r, "Z \otimes E \otimes \phi^{-1}_{a,-a}"'] & Z \otimes E
\end{tikzcd}$$

The top rectangle commutes by naturality of Δ in an additive category. The bottom triangle commutes by naturality of ∇ in an additive category. Finally, the remaining regions of the diagram commute by additivity of $- \otimes -$. By functoriality of $- \otimes -$, it follows that the left composition is $\widetilde{x + x'}$ and the right composition is $\widetilde{x} + \widetilde{x}'$, so they are equal as desired. Thus Ψ is a homomorphism of abelian groups, as desired.

Now, we claim that Ψ is an inverse to Φ_{Z,S^a}. Since Φ_{Z,S^a} and Ψ are homomorphisms it suffices to check that they are inverses on generators. First, let $x : S^b \to Z \otimes E \otimes S^a$ in $\pi_b(Z \otimes E \otimes S^a)$. We would like to show that $\Phi_{Z,S^a}(\Psi(x)) = x$. Consider the following diagram, where here we are passing to a symmetric strict monoidal category:

$$\begin{tikzcd}
S^b \arrow[r, "\cong"] \arrow[dd, "x"'] & S^b \otimes S^{-a} \otimes S^a \arrow[d, dashed, "x \otimes S^{-a} \otimes S^a"] \arrow[dr, "x \otimes S^{-a} \otimes e \otimes S^a"] & \\
& Z \otimes E \otimes S^a \otimes S^{-a} \otimes S^a \arrow[r, "Z \otimes E \otimes S^a \otimes S^{-a} \otimes e \otimes S^a"'] & Z \otimes E \otimes S^a \otimes S^{-a} \otimes E \otimes S^a \arrow[ddl, "Z \otimes E \otimes \phi^{-1}_{a,-a} \otimes E \otimes S^a"] \\
Z \otimes E \otimes S^a \arrow[ur, "Z \otimes E \otimes S^a \otimes \phi_{-a,a}"] \arrow[r, equal] & Z \otimes E \otimes S^a \arrow[u, dashed, "Z \otimes E \otimes \phi_{a,-a} \otimes S^a"'] \arrow[d, dashed, "Z \otimes E \otimes e \otimes S^a"] & \\
& Z \otimes E \otimes E \otimes S^a \arrow[ul, "Z \otimes \mu \otimes S^a"] &
\end{tikzcd}$$

The top left trapezoid commutes since the isomorphism $S^b \xrightarrow{\cong} S^b \otimes S^{-a} \otimes S^a$ may be given as $S^b \otimes \phi_{-a,a}$ (see Remark 2.4), in which case the trapezoid commmutes by functoriality of $- \otimes -$. The triangle below that commutes by coherence for the $\phi_{a,b}$'s. The bottom left triangle commutes by unitality for μ. The top right triangle commutes by functoriality of $- \otimes -$. Finally, the bottom right triangle commutes by functoriality of $-\otimes-$. It follows by unravelling definitions that the two outside compositions are x and $\Phi_{Z,S^a}(\Psi(x))$, so indeed we have $\Phi_{Z,S^a}(\Psi(x)) = x$ since the diagram commutes.

On the other hand, suppose we are given a homogeneous pure tensor $x\otimes y$ in $\pi_*(Z\otimes E)\otimes_{\pi_*(E)} \pi_*(E\otimes S^a)$, so $x : S^b \to Z\otimes E$ and $y : S^c \to E\otimes S^a$ for some $b, c \in A$. Then we would like to show that $\Psi(\Phi_{Z,S^a}(x\otimes y)) = x\otimes y$. Unravelling definitions, $\Psi(\Phi_{Z,S^a}(x\otimes y))$ is the homogeneous pure tensor $\widetilde{xy}\otimes\widetilde{e}$, where $\widetilde{e}$ is the map $e\otimes S^a : S^a \to E\otimes S^a$ is defined above, and by functoriality of $-\otimes-$, $\widetilde{xy} : S^{b+c-a} \to Z\otimes E$ is the composition

$$\begin{tikzcd}
S^{b+c-a} \arrow[d, "\cong"] \\
S^b\otimes S^c\otimes S^{-a} \arrow[d, "x\otimes y\otimes S^{-a}"] \\
Z\otimes E\otimes E\otimes S^a\otimes S^{-a} \arrow[d, "Z\otimes\mu\otimes S^a\otimes S^{-a}"] \\
Z\otimes E\otimes S^a\otimes S^{-a} \arrow[d, "Z\otimes E\otimes\phi^{-1}_{a,-a}"] \\
Z\otimes E
\end{tikzcd}$$

Now, define $r \in \pi_{c-a}(E)$ to be the composition

$$S^{c-a} \cong S^c\otimes S^{-a} \xrightarrow{y\otimes S^{-a}} E\otimes S^a\otimes S^{-a} \xrightarrow{E\otimes\phi^{-1}_{a,-a}} E.$$

First, we claim that $x\cdot r = \widetilde{xy}$. To that end, consider the following diagram, where here we are again passing to a symmetric strict monoidal category:

$$\begin{tikzcd}
S^{b+c-a} \arrow[r, "\cong"] & S^b\otimes S^c\otimes S^{-a} \arrow[r, "x\otimes y\otimes S^{-a}"] & Z\otimes E\otimes E\otimes S^a\otimes S^{-a} \arrow[r, "Z\otimes\mu\otimes S^a\otimes S^{-a}"] \arrow[d, "Z\otimes E\otimes E\otimes\phi^{-1}_{a,-a}"'] & Z\otimes E\otimes S^a\otimes S^{-a} \arrow[d, "Z\otimes E\otimes\phi^{-1}_{a,-a}"] \\
& & Z\otimes E\otimes E \arrow[r, "Z\otimes\mu"'] & Z\otimes E
\end{tikzcd}$$

Commutativity is functoriality of $-\otimes-$, which also tells us that the two outside compositions are $\widetilde{xy}$ (on top) and $x\cdot r$ (on the bottom), so they are equal as desired. On the other hand, we claim that $r\cdot\widetilde{e} = y$. To see this, consider the following diagram:

$$\begin{tikzcd}
S^c \arrow[rrr, "\cong"] \arrow[dd, "y"'] & & & S^c\otimes S^{-a}\otimes S^a \arrow[dll, "y\otimes S^{-a}\otimes S^a"'] \arrow[d, "y\otimes S^{-a}\otimes e\otimes S^a"] \\
& E\otimes S^a\otimes S^{-a}\otimes S^a \arrow[rr, "E\otimes S^a\otimes S^{-a}\otimes e\otimes S^a"'] \arrow[dl, "E\otimes S^a\otimes\phi^{-1}_{-a,a}"'] \arrow[d, "E\otimes\phi^{-1}_{a,-a}\otimes S^a"] & & E\otimes S^a\otimes S^{-a}\otimes E\otimes S^a \arrow[dd, "E\otimes\phi^{-1}_{a,-a}\otimes E\otimes S^a"] \\
E\otimes S^a \arrow[r, equal] & E\otimes S^a \arrow[drr, "E\otimes e\otimes S^a"] & & \\
E\otimes E\otimes S^a \arrow[u, "\mu\otimes S^a"] \arrow[rrr, equal] & & & E\otimes E\otimes S^a
\end{tikzcd}$$

By Remark 2.4, we may take the top arrow to be $S^c\otimes\phi_{-a,a}$, in which case the top left triangle commutes by functoriality of $-\otimes-$. The bottom trapezoid commutes by unitality of μ. Every other region commutes either by definition or by functoriality of $-\otimes-$. The top composition is

$r \cdot \widetilde{e}$, so we have shown $r \cdot \widetilde{e} = y$ as desired. Thus, we have that

$$\Psi(\Phi_{Z,S^a}(x \otimes y)) = \widetilde{xy} \otimes \widetilde{e} = x \cdot r \otimes \widetilde{e} = x \otimes r \cdot \widetilde{e} = x \otimes y,$$

as desired. Hence we have shown Ψ is both a left and right inverse for Φ_{Z,S^a}, so that indeed S^a belongs to $\mathcal{E}$ as desired.

Now, we would like to show that given a distinguished triangle in $\mathcal{SH}$

$$X \xrightarrow{f} Y \xrightarrow{g} W \xrightarrow{h} \Sigma X,$$

if two of three of the objects X, Y, and W belong to $\mathcal{E}$, then so does the third. From now on, write L^E_* to denote the functor from $\mathcal{SH}$ to A-graded abelian groups sending $X \mapsto \pi_*(Z \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes X)$. Then $\Phi_{Z,-}$ is a natural transformation $L^E_* \Rightarrow \pi_*(Z \otimes E \otimes -) = Z_*(E \otimes -)$. First, recall that it follows generally that in an adjointly triangulated category (Definition 2.8), which $\mathcal{SH}$ is by Proposition 2.7, given a distinguished triangle (f, g, h) we have a long exact sequence (see Definition A.1 for the definition of an exact sequence in an additive category, and see Proposition A.9 for the explicit contruction of the LES associated to a distinguished triangle in an adjointly triangulated category):

$$\Omega Y \xrightarrow{\Omega g} \Omega W \xrightarrow{\widetilde{h}} X \xrightarrow{f} Y \xrightarrow{g} W \xrightarrow{h} \Sigma X \xrightarrow{\Sigma f} \Sigma Y,$$

where $\widetilde{h} : \Omega W \to X$ is the adjoint of $h : W \to \Sigma X$. Then since $\mathcal{SH}$ is further a tensor triangulated category (Definition 2.2), we have that the above sequence remains exact even after tensoring by E on the left (see Proposition A.11 for details), so we have the following exact sequence in $\mathcal{SH}$:

$$E \otimes \Omega Y \xrightarrow{E\otimes\Omega g} E \otimes \Omega W \xrightarrow{E\otimes\widetilde{h}} E \otimes X \xrightarrow{E\otimes f} E \otimes Y \xrightarrow{E\otimes g} E \otimes W \xrightarrow{E\otimes h} E \otimes \Sigma X \xrightarrow{E\otimes\Sigma f} E \otimes \Sigma Y.$$

We can then apply $[S^*, -] = \pi_*(-)$ to it, which yields the following exact sequence of A-graded abelian groups:

$$E_*(\Omega Y) \xrightarrow{E_*(\Omega g)} E_*(\Omega W) \xrightarrow{E_*(\widetilde{h})} E_*(X) \xrightarrow{E_*(f)} E_*(Y) \xrightarrow{E_*(g)} E_*(W) \xrightarrow{E_*(h)} E_*(\Sigma X) \xrightarrow{E_*(f)} E_*(\Sigma Y).$$

Now, we can tensor this sequence with $\pi_*(Z \otimes E)$ on the left over $\pi_*(E)$, and since $\pi_*(Z \otimes E)$ is a flat right $\pi_*(E)$ module, we get that the top row in the following diagram is exact:

$$\begin{tikzcd}
L^E_*(\Omega Y) \arrow[r, "L^E_*(\Omega g)"] \arrow[d, "\Phi_{Z,\Omega Y}"'] & L^E_*(\Omega W) \arrow[r, "L^E_*(\widetilde{h})"] \arrow[d, "\Phi_{Z,\Omega W}"'] & L^E_*(X) \arrow[r, "L^E_*(f)"] \arrow[d, "\Phi_{Z,X}"'] & L^E_*(Y) \arrow[r, "L^E_*(g)"] \arrow[d, "\Phi_{Z,Y}"'] & L^E_*(W) \arrow[r, "L^E_*(h)"] \arrow[d, "\Phi_{Z,W}"'] & L^E_*(\Sigma X) \arrow[r, "L^E_*(\Sigma f)"] \arrow[d, "\Phi_{Z,\Sigma X}"'] & L^E_*(\Sigma Y) \arrow[d, "\Phi_{Z,\Sigma Y}"'] \\
Z_*(E \otimes \Omega Y) \arrow[r, "Z_*(E\otimes\Omega g)"'] & Z_*(E \otimes \Omega W) \arrow[r, "Z_*(E\otimes h)"'] & Z_*(E \otimes X) \arrow[r, "Z_*(E\otimes f)"'] & Z_*(E \otimes Y) \arrow[r, "Z_*(E\otimes g)"'] & Z_*(E \otimes W) \arrow[r, "Z_*(E\otimes h)"'] & Z_*(E \otimes \Sigma X) \arrow[r, "Z_*(E\otimes\Sigma f)"'] & Z_*(E \otimes \Sigma Y)
\end{tikzcd}$$

This diagram further commutes by naturality of $\Phi_{Z,-}$. Now, supposing that two of three of X, Y, and W belong to $\mathcal{E}$, by Lemma 5.5, if $\Phi_{Z,V}$ is an isomorphism for some object V in $\mathcal{SH}$ then $\Phi_{Z,\Omega V}$ and $\Phi_{Z,\Sigma V}$ are. Thus by the five lemma, it follows that the middle three vertical arrows in the above diagram are necessarily all isomorphisms if any two of them are, so we have shown that $\mathcal{E}$ is closed under two-of-three for exact triangles, as desired.

Finally, it remains to show that $\mathcal{E}$ is closed under arbitrary coproducts. Let $\{W_i\}_{i\in I}$ be a collection of objects in $\mathcal{E}$ indexed by some set I. Then we'd like to show that $W := \bigoplus_i W_i$ belongs to $\mathcal{E}$. First of all, note that $- \otimes -$ preserves arbitrary coproducts in each argument, as it has a right adjoint $F(-, -)$. Thus without loss of generality, given any object X in $\mathcal{SH}$, we may take $\bigoplus_i X \otimes W_i = X \otimes \bigoplus_i W_i$ (as $X \otimes \bigoplus_i W_i$ *is* a coproduct of all the $X \otimes W_i$'s). Now, recall that we have chosen each S^a to be a compact object (Definition 2.5), so that given any object X and collection of objects $\{Y_i\}_{i\in I}$ in $\mathcal{SH}$, if $Y := \bigoplus_{i\in I} Y_i$, then the canonical map

$$q_{X,Y_i} : \bigoplus_i X_*(Y_i) = \bigoplus_i [S^*, X \otimes Y_i] \to [S^*, \bigoplus_i X \otimes Y_i] = [S^*, X \otimes Y] = X_*(Y)$$

is an isomorphism, natural in Y_i for each i. Note in particular that q_{E,W_i} is an isomorphism of left $\pi_*(E)$-modules. To see this, first note by additivity of q_{E,W_i}, it suffices to check that $q_{E,W_i}(r \cdot x) = r \cdot q_{E,W_i}(x)$ for each homogeneous $r \in \pi_*(E)$ and homogeneous $x \in E_*(W_i)$ for some i, as such x generate $\bigoplus_i E_*(W_i)$ by definition. Then given $r : S^a \to E$ and $x : S^b \to E \otimes W_i$, consider the following diagram

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] & S^a \otimes S^b \arrow[r, "x\otimes y"] & E \otimes E \otimes W_i \arrow[rr, "E\otimes\iota_{E\otimes W_i}"] \arrow[ddd, "\mu\otimes W_i"'] \arrow[drr, "E\otimes E\otimes\iota_{W_i}"] & & E \otimes \bigoplus_i (E \otimes W_i) \arrow[d, equal] \\
& & & & E \otimes E \otimes W \arrow[d, "\mu\otimes W"] \\
& & & & E \otimes W \arrow[d, equal] \\
& & E \otimes W_i \arrow[rr, "\iota_{E\otimes W_i}"'] \arrow[urr, "E\otimes\iota_{W_i}"] & & \bigoplus_i (E \otimes W_i)
\end{tikzcd}$$

where $\iota_{E\otimes W_i} : E \otimes W_i \hookrightarrow \bigoplus_i (E \otimes W_i)$ and $\iota_{W_i} : W_i \hookrightarrow \bigoplus_i W_i$ are the maps determined by the definition of the coproduct. Commutativity of the two triangles is by the fact that $E \otimes -$ is colimit preserving. Commutativity of the trapezoid is functoriality of $- \otimes -$. Thus, since q_{E,W_i} is a homomorphism of left A-graded $\pi_*(E)$-modules, the top right arrow in the following diagram is well-defined:

$$\begin{tikzcd}
\bigoplus_i Z_*(E) \otimes_{\pi_*(E)} E_*(W_i) \arrow[r, equal] \arrow[d, "\bigoplus_i \Phi_{Z,W_i}"'] & Z_*(E) \otimes_{\pi_*(E)} \bigoplus_i E_*(W_i) \arrow[r, "Z_*(E)\otimes_{\pi_*(E)} q_{E,W_i}"] & Z_*(E) \otimes_{\pi_*(E)} E_*(W) \arrow[d, "\Phi_{Z,W}"] \\
\bigoplus_i Z_*(E \otimes W_i) \arrow[r, "q_{Z,E\otimes W_i}"] & Z_*(\bigoplus_i E \otimes W_i) \arrow[r, equal] & Z_*(E \otimes W)
\end{tikzcd} \tag{5}$$

We wish to show this diagram commutes. Again, since each map here is a homomorphism, it suffices to chase generators. By definition, a generator of the top left element is a homogeneous pure tensor in $E_*(E) \otimes_{\pi_*(E)} E_*(W_i)$ for some i in I. Given classes $x : S^a \to Z \otimes E$ in $Z_*(E)$ and $y : S^b \to E \otimes W_i$ in $E_*(W_i)$, consider the following diagram:

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] & S^a \otimes S^b \arrow[r, "x\otimes y"] & Z \otimes E \otimes E \otimes W_i \arrow[r, "Z\otimes E\otimes\iota_{E\otimes W_i}"] \arrow[d, "Z\otimes\mu\otimes W_i"'] \arrow[dr, "Z\otimes E\otimes E\otimes\iota_{W_i}"] & Z \otimes E \otimes \bigoplus_i E \otimes W_i \arrow[d, equal] \\
& & Z \otimes E \otimes W_i \arrow[d, "\iota_{Z\otimes E\otimes W_i}"'] \arrow[dr, "Z\otimes E\otimes\iota_{W_i}"] & Z \otimes E \otimes E \otimes W \arrow[d, "Z\otimes\mu\otimes W"] \\
& & \bigoplus_i Z \otimes E \otimes W_i \arrow[r, equal] & Z \otimes E \otimes W
\end{tikzcd}$$

Unravelling definitions, the two outside compositions are the two ways to chase $x \otimes y$ around diagram (5). The two triangles commute again by the fact that $- \otimes -$ preserves colimits in each argument. Commutativity of the inner parallelogram is functoriality of $- \otimes -$. Thus diagram (5) tells us $\Phi_{Z,W}$ is an isomorphism, since q_{E,W_i} and $q_{Z,E\otimes W_i}$ are isomorphisms, and Φ_{Z,W_i} is an isomorphism for each i in I, meaning $\bigoplus_i \Phi_{W_i}$ is as well.

Thus, we've shown the class $\mathcal{E}$ of objects W for which $\Phi_{Z,W}$ is an isomorphism contains the S^a's, is closed under two-of-three for distinguished triangles, and is closed under arbitrary coproducts. Thus, it follows that $\mathcal{E}$ contains the class of all cellular objects in $\mathcal{SH}$, as desired.

Now, suppose that $\pi_*(E \otimes W)$ is a flat left $\pi_*(E)$-module, then we'd like to show $\Phi_{Z,W}$ is an isomorphism for all cellular Z in $\mathcal{SH}$. Showing this is entirely analagous to above, so we only outline the argument. Let $\mathcal{E}$ be the class of Z in $\mathcal{SH}$ such that $\Phi_{Z,W}$ is an isomorphism. Then in order to show $\mathcal{E}$ contains every cellular object, it suffices to show it contains the S^a's, is closed under two-of-three for distinguished triangles, and is closed under arbitrary coproducts.

To see $\mathcal{E}$ contains the S^a's, consider the map

$$\Psi : \pi_*(S^a \otimes E \otimes W) \to \pi_*(S^a \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W)$$

sending $x : S^b \to S^a \otimes E \otimes W$ to $\widetilde{e} \otimes \widetilde{x}$, where $\widetilde{e} \in \pi_a(S^a \otimes E)$ is the map $S^a \otimes e : S^a \to S^a \otimes E$, and $\widetilde{x} \in \pi_{b-a}(E \otimes W)$ is the map

$$\widetilde{x} : S^{b-a} \xrightarrow{\phi_{-a,b}} S^{-a} \otimes S^b \xrightarrow{S^{-a} \otimes x} S^{-a} \otimes S^a \otimes E \otimes W \xrightarrow{\phi_{-a,a}^{-1} \otimes E \otimes W} E \otimes W.$$

Then checking that Ψ is a left and right inverse to $\Phi_{S^a,W}$ is entirely analagous, so that S^a belongs to $\mathcal{E}$ as desired.

To see $\mathcal{E}$ is closed under two-of-three for distinguished triangles, let

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

be a distinguished triangle in $\mathcal{SH}$. Then an analagous argument as above (using Proposition A.9 and Proposition A.11) yields a long exact sequence of A-graded abelian groups

$$\begin{tikzcd}
& \pi_*(\Omega Y \otimes E) \arrow[r, "\pi_*(\Omega g \otimes E)"] & \pi_*(\Omega Z \otimes E) \arrow[dll, "\pi_*(\widetilde{h} \otimes E)"'] \\
\pi_*(X \otimes E) \arrow[r, "\pi_*(f \otimes E)"'] & \pi_*(Y \otimes E) \arrow[r, "\pi_*(g \otimes E)"] & \pi_*(Z \otimes E) \arrow[dll, "\pi_*(h \otimes E)"'] \\
\pi_*(\Sigma X \otimes E) \arrow[r, "\pi_*(\Sigma f \otimes E)"'] & \pi_*(\Sigma Y \otimes E) &
\end{tikzcd}$$

Then since $\pi_*(E \otimes W)$ is a flat left $\pi_*(E)$-module, we can tensor the above long exact sequence with $\pi_*(E \otimes W)$ on the right to obtain a long exact sequence which fits in the left column of the following commuting diagram:

$$\begin{tikzcd}
R^E_*(\Omega Y) \arrow[r, "\Phi_{\Omega Y,W}"] \arrow[d, "R^E_*(\Omega g)"'] & \pi_*(\Omega Y \otimes E \otimes W) \arrow[d, "\pi_*(\Omega g \otimes E \otimes W)"] \\
R^E_*(\Omega Z) \arrow[r, "\Phi_{\Omega Z,W}"] \arrow[d, "R^E_*(\widetilde{h})"'] & \pi_*(\Omega Z \otimes E \otimes W) \arrow[d, "\pi_*(\widetilde{h} \otimes E \otimes W)"] \\
R^E_*(X) \arrow[r, "\Phi_{X,W}"] \arrow[d, "R^E_*(f)"'] & \pi_*(X \otimes E \otimes W) \arrow[d, "\pi_*(f \otimes E \otimes W)"] \\
R^E_*(Y) \arrow[r, "\Phi_{Y,W}"] \arrow[d, "R^E_*(g)"'] & \pi_*(Y \otimes E \otimes W) \arrow[d, "\pi_*(g \otimes E \otimes W)"] \\
R^E_*(Z) \arrow[r, "\Phi_{Z,W}"] \arrow[d, "R^E_*(h)"'] & \pi_*(Z \otimes E \otimes W) \arrow[d, "\pi_*(h \otimes E \otimes W)"] \\
R^E_*(\Sigma X) \arrow[r, "\Phi_{\Sigma X,W}"] \arrow[d, "R^E_*(\Sigma f)"'] & \pi_*(\Sigma X \otimes E \otimes W) \arrow[d, "\pi_*(\Sigma f \otimes E \otimes W)"] \\
R^E_*(\Sigma Y) \arrow[r, "\Phi_{\Sigma Y,W}"] & \pi_*(\Sigma Y \otimes E \otimes W)
\end{tikzcd}$$

where R^E_* denotes the functor from $\mathcal{SH}$ to A-graded abelian groups sending $X \mapsto \pi_*(X \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W)$, so that $\Phi_{-,W}$ is a natural homomorphism $R^E_*(-) \Rightarrow \pi_*(- \otimes E \otimes W)$. Then finally by Lemma 5.5 and the five lemma, if any two of three of the middle three horizontal arrows are isomorphisms, then all three of the horizontal arrows are isomorphisms, as desired.

Finally, in order to show $\mathcal{E}$ is closed under arbitrary coproducts, suppose we have a collection of objects $\{Z_i\}_{i\in I}$ in $\mathcal{E}$ indexed by some (small) set $\mathcal{E}$. Then we'd like to show $Z := \bigoplus_{i\in I} Z_i$ also belongs to $\mathcal{E}$. First note that since the S^a's are compact, for any object Y we have isomorphisms

$$q_{Z_i,Y} : \bigoplus_i Z_{i*}(Y) = \bigoplus_i [S^*, Z_i \otimes Y] \to [S^*, \bigoplus_i (Z_i \otimes Y)] = [S^*, Z \otimes Y] = Z_*(Y).$$

It is straightforward to verify that $q_{Z_i,E} : \bigoplus_i Z_{i*}(E) \to Z_*(E)$ is not only an isomorphism of abelian groups, but an isomorphism of right A-graded $\pi_*(E)$-modules, so that the top arrow in the following diagram is well-defined:

$$\begin{tikzcd}
\bigoplus_i \big(Z_{i*}(E) \otimes_{\pi_*(E)} E_*(W)\big) \arrow[r, equal] \arrow[d, "\bigoplus_i \Phi_{Z_i,W}"'] & \bigoplus_i (Z_{i*}(E)) \otimes_{\pi_*(E)} E_*(W) \arrow[r, "q_{Z_i,E}\otimes E_*(W_i)"] & Z_*(E) \otimes_{\pi_*(E)} E_*(W) \arrow[d, "\Phi_{Z,W}"] \\
\bigoplus_i Z_{i*}(E \otimes W) \arrow[rr, "q_{Z_i,E\otimes W}"] & & Z_*(E \otimes W)
\end{tikzcd}$$

Then a simple diagram chase yields the diagram commutes, so that $\Phi_{Z,W}$ is an isomorphism, assuming all the $\Phi_{Z_i,W}$'s are. □

5.2. **Modules over monoid objects in $\mathcal{SH}$.** Now, before we prove our next theorem (an analog of the universal coefficient theorem in $\mathcal{SH}$), we need to develop some of the theory of (left) module objects over monoid objects in $\mathcal{SH}$. For a review of the basic definitions and properties of module objects over monoid objects in symmetric monoidal categories, see Appendix C.2. Recall specifically that given a monoid object (E, μ, e) in $\mathcal{SH}$, the category E-**Mod** of (left) E-module objects is additive (Proposition C.15), and the forgetful functor E-**Mod** $\to \mathcal{SH}$ preserves arbitrary coproducts and has a right adjoint $\mathcal{SH} \to E$-**Mod** taking an object X in $\mathcal{SH}$ to the *free E-module* $E \otimes X$ (Proposition C.12).

Proposition 5.7. *Let (E, μ, e) be a monoid object in $\mathcal{SH}$. Then the assignment $\pi_* : (N, \kappa) \mapsto \pi_*(N)$ yields an additive functor from E-**Mod** to the category $\pi_*(E)$-$\mathbf{Mod}^A$ of A-graded left $\pi_*(E)$-modules and degree-preserving homomorphisms between them, and in fact, it preserves arbitrary coproducts. In particular, if (N, κ) is an E-module object in $\mathcal{SH}$, then we view it with its* canonical *A-graded left $\pi_*(E)$-module structure given by the graded map*

$$\pi_*(E) \times \pi_*(N) \to \pi_*(N)$$

sending a class $r : S^a \to E$ and $x : S^b \to N$ to the composition

$$r \cdot x : S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{r\otimes x} E \otimes N \xrightarrow{\kappa} N.$$

Proof. First let (N, κ) be an E-module object. Let $a, b, c \in A$ and $x, x' : S^a \to N$, $y : S^b \to E$, and $z, z' \in S^c \to E$. Then by Lemma B.9, it suffices to show that

(1) $y \cdot (x + x') = y \cdot x + y \cdot x'$,
(2) $(z + z') \cdot x = z \cdot x + z' \cdot x$,
(3) $(zy) \cdot x = z \cdot (y \cdot x)$,
(4) $e \cdot x = x$.

The first two axioms follow by Lemma 4.2. To see (3), consider the diagram:

$$\begin{tikzcd}
 & & & E \otimes N \arrow[d, "\kappa"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^c \otimes S^b \otimes S^a \arrow[r, "z \otimes y \otimes x"] & E \otimes E \otimes N \arrow[ur, "E \otimes \kappa"] \arrow[dr, "\mu \otimes N"'] & N \\
 & & & E \otimes N \arrow[u, "\kappa"']
\end{tikzcd}$$

It commutes by coherence for κ. By functoriality of $- \otimes -$, the two outside compositions equal $z \cdot (y \cdot x)$ on the top and $(z \cdot y) \cdot x$ on the bottom. Hence, they are equal, as desired.

Next, to see (4), consider the following diagram:

$$\begin{tikzcd}
S^a \arrow[rr, "x"] \arrow[dr, "x"'] \arrow[ddr, bend right, "e \otimes x"'] & & N \\
 & N \arrow[ur, equals] \arrow[d, dashed, "e \otimes N"] & \\
 & E \otimes N \arrow[uur, bend right, "\kappa"'] &
\end{tikzcd}$$

The top triangle commutes by definition. The left triangle commutes by functoriality of $- \otimes -$. The right triangle commutes by unitality of κ. The top composition is x while the bottom is $e \cdot x$, thus they are necessarily equal since the diagram commutes.

Now, we'd like to show that if $f : (N, \kappa) \to (N', \kappa)$ is a homomorphism of E-module objects, then $\pi_*(f) : \pi_*(N) \to \pi_*(N')$ is a homomorphism of left $\pi_*(E)$-modules. To see this, let $r : S^a \to E$ in $\pi_a(E)$ and $x, x : S^b \to N$ in $\pi_b(N)$. We'd like to show that $\pi_*(f)(x + x') = \pi_*(f)(x) + \pi_*(f)(x')$ and $\pi_*(f)(r \cdot x) = r \cdot \pi_*(f)(x)$. To see the former, consider the following diagram:

$$\begin{tikzcd}
 & & & N' \oplus N' \arrow[d, "\nabla"] \\
S^a \arrow[r, "\Delta"] & S^a \oplus S^a \arrow[r, "x \oplus x'"] & N \oplus N \arrow[ur, "f \oplus f"] \arrow[dr, "\nabla"'] & N' \\
 & & & N \arrow[u, "f"']
\end{tikzcd}$$

It commutes by naturality of ∇ in an additive category. The top composition is $\pi_*(f)(x) + \pi_*(f)(x')$, while the bottom is $\pi_*(f)(x+x')$, so they are equal as desired. To see that $\pi_*(f)(r \cdot x) = r \cdot \pi_*(f)(x)$, consider the following diagram:

$$\begin{tikzcd}
 & & & E \otimes N' \arrow[d, "\kappa'"] \\
S^{a+b} \arrow[r, "\phi_{b,a}"] & S^b \otimes S^a \arrow[r, "r \otimes x"] & E \otimes N \arrow[ur, "E \otimes f"] \arrow[dr, "\kappa"'] & N' \\
 & & & N \arrow[u, "f"']
\end{tikzcd}$$

It commutes by the fact that f is a homomorphism of E-module objects. The bottom composition is $\pi_*(f)(r \cdot x)$, while the top composition is $r \cdot \pi_*(f)(x)$, so they are equal, as desired.

Next we claim this preserves arbitrary coproducts. First of all, note that $\pi_*(0) = [S^*, 0] = 0$ by definition, since 0 is terminal. Now suppose we have a family of objects $(N_i, \kappa_i) \in E$-**Mod** then we would like to show that there is an degree-preserving isomorphism of A-graded left $\pi_*(E)$-modules $\bigoplus_i \pi_*(N_i) \xrightarrow{\cong} \pi_*(\bigoplus_i N_i)$ such that the following diagram commutes for all i:

$$\begin{tikzcd} \pi_*(N_i) \arrow[d, "\iota_{\pi_*(N_i)}"'] \arrow[dr, "\pi_*(\iota_{N_i})"] & \\ \bigoplus_i \pi_*(N_i) \arrow[r, "\cong"] & \pi_*(\bigoplus_i N_i) \end{tikzcd} \tag{6}$$

First of all, since each S^a is compact, for all $a \in A$ we have isomorphisms

$$\bigoplus_i \pi_a(N_i) = \bigoplus_i [S^a, N_i] \xrightarrow{\cong} [S^a, \bigoplus_i N_i] = \pi_a(\bigoplus_i N_i),$$

and these combine together to yield A-graded isomorphisms $q_{\{N_i\}} : \bigoplus_i \pi_*(N_i) \xrightarrow{\cong} \pi_*(\bigoplus_i N_i)$. Explicitly unravelling defintions, the above maps send a generator $x : S^a \to N_i$ in $\pi_a(N_i)$ to the class $S^a \xrightarrow{x} N_i \xrightarrow{\iota_{N_i}} \bigoplus_i N_i$. To see this isomorphism is further an isomorphism of left $\pi_*(E)$-modules, it suffices to show that given a generator $r : S^a \to E$ in $\pi_*(E)$ and a generator $x : S^b \to N_i$ in $\pi_b(N_i) \leq \bigoplus_i \pi_*(N_i)$, that $r \cdot q_{\{N_i\}}(x) = q_{\{N_i\}}(r \cdot x)$. To that end, consider the following diagram:

$$\begin{tikzcd} S^{a+b} \arrow[r, "\phi_{a,b}"] & S^a \otimes S^b \arrow[r, "r\otimes x"] & E \otimes N_i \arrow[r, "E\otimes\iota_{N_i}"] \arrow[dd, "\kappa_i"] & E \otimes \bigoplus_i N_i \arrow[d, "\cong"] \\ & & & \bigoplus_i (E \otimes N_i) \arrow[d, "\bigoplus_i \kappa_i"] \\ & & N_i \arrow[r, "\iota_{N_i}"] & \bigoplus_i N_i \end{tikzcd}$$

Unravelling definitions, the top composition is $r \cdot q_{\{N_i\}}(x)$, while the bottom composition is $q_{\{N_i\}}(r \cdot x)$. In Proposition C.14, we showed that $\iota_{N_i} : N_i \hookrightarrow \bigoplus_i N_i$ is an E-module object homomorphism, and the right map is precisely the action map for $\bigoplus_i N_i$ as an E-module object, so the diagram does indeed commute — $q_{\{N_i\}}$ is a degree-preserving isomorphism of A-graded left $\pi_*(E)$-module objects, as desired. Finally, to see diagram (6) commutes, observe that by unravelling definitions, given a homogeneous element $x : S^a \to N_i$ in $\pi_*(N_i)$, chasing it either way around the diagram yields the composition

$$S^a \xrightarrow{x} N_i \xrightarrow{\iota_{N_i}} \bigoplus_i N_i,$$

so that diagram (6) commutes for generators, and thus commutes entirely, since all the maps involved are homomorphisms. □

Remark 5.8. In the above proposition, we have shown that given an E-module object (N, κ) in $\mathcal{SH}$, $\pi_*(N)$ is canonically an A-graded left $\pi_*(E)$-module. In particular, we may apply this proposition to the free E-module $E \otimes X$ (Proposition C.12). It is straightforward to see, and we leave it to the reader to check, that the A-graded left $\pi_*(E)$-module structure on $E_*(X) = \pi_*(E \otimes X)$ induced by the above proposition is precisely the canonical module structure from Proposition 4.3. In fact, the above proposition entirely subsumes the first half of Proposition 4.3 (although we give the two separate statements for the sake of clarity). Thus, there continues to be no ambiguity when talking about the left $\pi_*(E)$-module structure on $E_*(X)$.

Lemma 5.9. *Let (E, μ, e) be a monoid object in $\mathcal{SH}$, and suppose (N, κ) is a module object over E (Definition C.8). Then for all $a \in A$, the a^{th} suspension $\Sigma^a N$ of N is canonically an E-module* *object, with action map given by*

$$\kappa^a : E \otimes \Sigma^a N = E \otimes S^a \otimes N \xrightarrow{\tau \otimes N} S^a \otimes E \otimes N \xrightarrow{S^a \otimes \kappa} S^a \otimes N = \Sigma^a N.$$

Furthermore, given an E-module homomorphism $f : (N, \kappa) \to (N', \kappa')$, $\Sigma^a f : \Sigma^a N \to \Sigma^a N'$ is likewise an E-module homomorphism.

Proof. In this proof, we are assuming that unitality and associativity hold up to strict equality, by the coherence theorem for monoidal categories. In order to show $(\Sigma^a N, \kappa^a)$ is a module object over E, we need to show κ^a makes the two coherence diagrams in Definition C.8 commute. First, to see the first diagram commutes, consider the following diagram:

$$\begin{tikzcd}
S^a \otimes N \arrow[r, "e \otimes S^a \otimes N"] \arrow[dr, "S^a \otimes e \otimes N"'] \arrow[ddr, equal] & E \otimes S^a \otimes N \arrow[d, "\tau \otimes N"] \\
& S^a \otimes E \otimes N \arrow[d, "S^a \otimes \kappa"] \\
& S^a \otimes N
\end{tikzcd}$$

The top inner triangle commutes by coherence for a symmetric monoidal category, and the bottom inner triangle commutes by the coherence condition for κ. To see the other module condition for $\widetilde{\kappa}$, consider the following diagram:

$$\begin{tikzcd}
E \otimes E \otimes S^a \otimes N \arrow[rr, "\mu \otimes S^a \otimes N"] \arrow[d, "E \otimes \tau \otimes N"'] \arrow[dr, "\tau_{E \otimes E, S^a} \otimes N"] & & E \otimes S^a \otimes N \arrow[d, "\tau \otimes N"] \\
E \otimes S^a \otimes E \otimes N \arrow[r, "\tau \otimes E \otimes N"'] \arrow[d, "E \otimes S^a \otimes \kappa"'] & S^a \otimes E \otimes E \otimes N \arrow[r, "S^a \otimes \mu \otimes N"] \arrow[d, dashed, "S^a \otimes E \otimes \kappa"] & S^a \otimes E \otimes N \arrow[d, "S^a \otimes \kappa"] \\
E \otimes S^a \otimes N \arrow[r, "\tau \otimes N"] & S^a \otimes E \otimes N \arrow[r, "S^a \otimes \kappa"] & S^a \otimes N
\end{tikzcd}$$

The top left triangle commutes by coherence for a symmetric monoidal category. The bottom left rectangle and top right trapezoid commute by naturality of τ. Finally, the bottom right square commutes by the coherence condition for κ.

Thus, we have shown that $\Sigma^a N$ is indeed an object in E-**Mod**, as desired. Now let $f : (N, \kappa) \to (N', \kappa')$ be a morphism in E-**Mod**, we would like to show $\Sigma^a f : \Sigma^a N \to \Sigma^a N'$ is also a homomorphism of E-modules. To that end, consider the following diagram:

$$\begin{tikzcd}
E \otimes S^a \otimes N \arrow[r, "E \otimes S^a \otimes f"] \arrow[d, "\tau \otimes N"'] & E \otimes S^a \otimes N' \arrow[d, "\tau \otimes N'"] \\
S^a \otimes E \otimes N \arrow[r, "S^a \otimes E \otimes f"] \arrow[d, "S^a \otimes \kappa"'] & S^a \otimes E \otimes N' \arrow[d, "S^a \otimes \kappa'"] \\
S^a \otimes N \arrow[r, "S^a \otimes f"'] & S^a \otimes N'
\end{tikzcd}$$

The top rectangle commutes by functoriality of $- \otimes -$, while the bottom commutes since f is an E-module homomorphism. Thus, $S^a \otimes f = \Sigma^a f$ is an E-module homomorphism, as desired. $\square$

Definition 5.10. We can extend the hom-groups in E-**Mod** (which is additive by Proposition C.15) to A-graded abelian groups $\mathrm{Hom}^*_{E\text{-}\mathbf{Mod}}(N, N')$ defined by

$$\mathrm{Hom}^a_{E\text{-}\mathbf{Mod}}(N, N') := \mathrm{Hom}_E(\Sigma^a N, N'),$$

where $\Sigma^a N$ is considered as an E-module object by the above lemma.

Lemma 5.11. *Given a monoid object (E,μ,e) in $\mathcal{SH}$, an object X in $\mathcal{SH}$, and some $a \in A$, the suspension of the free module $\Sigma^a(E \otimes X)$ is naturally isomorphic as an E-module object to the free E-module $E \otimes \Sigma^a X$.*

Proof. It suffices to show the map $S^a \otimes E \otimes X \xrightarrow{\tau\otimes X} E \otimes S^a \otimes X$ is a homomorphism of E-module objects, as we know it is an isomorphism and natural in X. To that end, consider the following diagram:

$$\begin{tikzcd}
E \otimes S^a \otimes E \otimes X \arrow[rr, "E\otimes\tau\otimes X"] \arrow[d, "\tau\otimes E\otimes X"'] & & E \otimes E \otimes S^a \otimes X \arrow[dd, "\mu\otimes S^a\otimes X"] \\
S^a \otimes E \otimes E \otimes X \arrow[urr, "\tau_{S^a, E\otimes E}\otimes X"'] \arrow[d, "S^a\otimes\mu\otimes X"'] & & \\
S^a \otimes E \otimes X \arrow[rr, "\tau\otimes X"] & & E \otimes S^a \otimes X
\end{tikzcd}$$

The top triangle commutes by coherence for a symmetric monoidal category. The bottom trapezoid commutes by naturality of τ. $\square$

Lemma 5.12. *Let (E,μ,e) be a monoid object in $\mathcal{SH}$, and suppose we have a collection of objects (N_i,κ_i) in E-**Mod**. Then for all $a \in A$, since Σ^a has a right adjoint Σ^{-a} (Proposition 2.7), it preserves coproducts in $\mathcal{SH}$, (which are coproducts in E-**Mod** by Proposition C.14), so we have an isomorphism*

$$\Sigma^a \bigoplus_i N_i \cong \bigoplus_i \Sigma^a N_i.$$

Then this isomorphism is an E-module homomorphism.

Proof. Consider the following diagram:

$$\begin{tikzcd}
E \otimes S^a \otimes \bigoplus_i N_i \arrow[r, "E\otimes\cong"] \arrow[d, "\tau\otimes\bigoplus_i N_i"'] \arrow[dr, "\cong"] & E \otimes \bigoplus_i (S^a \otimes N_i) \arrow[d, "\cong"] \\
S^a \otimes E \otimes \bigoplus_i N_i \arrow[d, "S^a\otimes\cong"'] & \bigoplus_i (E \otimes S^a \otimes N_i) \arrow[d, "\bigoplus_i(\tau\otimes N_i)"] \\
S^a \otimes \bigoplus_i (E \otimes N_i) \arrow[r, "\cong"] \arrow[d, "S^a\otimes\bigoplus_i \kappa_i"'] & \bigoplus_i (S^a \otimes E \otimes N_i) \arrow[d, "\bigoplus_i(S^a\otimes\kappa_i)"] \\
S^a \otimes \bigoplus_i N_i \arrow[r, "\cong"] & \bigoplus_i (S^a \otimes N_i)
\end{tikzcd}$$

The top region comutes by additivity of $E \otimes -$. The bottom two region commute by naturality of the additivity isomorphisms for $- \otimes -$. $\square$

5.3. A universal coefficient theorem.

Finally, we have the ingredients required to state and prove the following universal coefficient theorem:

Theorem 5.13. *Let (E,μ,e) be a monoid object and let X and Y be objects in $\mathcal{SH}$. Then if E and X are cellular and $E_*(X)$ is a graded projective (Definition B.15) left $\pi_*(E)$-module (via Proposition 4.3), then the map*

$$[X, E\otimes Y] \to \mathrm{Hom}_{\pi_*(E)}(E_*(X), E_*(Y)), \qquad [X \xrightarrow{f} E\otimes Y] \mapsto [\pi_*(\mu\otimes Y)\circ E_*(f)]$$

is an isomorphism, and extends to an A-graded isomorphism

$$[X, E\otimes Y]_* \to \mathrm{Hom}^*_{\pi_*(E)}(E_*(X), E_*(Y)).$$

Proof. Since: (1) $E \otimes X$ is a free E-module object (Proposition C.12), (2) $E_*(X) = \pi_*(E \otimes X)$ is a graded projective left $\pi_*(E)$-module, and (3) E and $E \otimes X$ are cellular (by Lemma 3.4), by Proposition 5.14 below it follows that $E \otimes X$ is a retract of $\bigoplus_i (E \otimes S^{a_i})$ in E-**Mod** for some collection of $a_i \in A$ indexed by some set I. Thus the desired result follows by Proposition 5.16 below with $N = E \otimes Y$ (which is considered as a free E-module by Proposition C.12). ☐

In the case that $Y = S$, this theorem becomes the more familiar statement:

$$E^*(X) \cong [X, E]_{-*} \cong [X, E \otimes S]_{-*} \cong \mathrm{Hom}^{-*}_{\pi_*(E)}(E_*(X), \pi_*(E)),$$

i.e., the E-cohomology of X is isomorphic to the dual of the E-homology of X when $E_*(X)$ is a graded projective module. Hence why we call it the universal coefficient theorem. The condition that $E_*(X)$ be graded projective is a rather technical one, although thankfully it is often satisfied in practice. In the case $\pi_*(E)$ is a field, or more generally a product of fields, every $\pi_*(E)$-module is projective and this is trivially satisified. Alternatively, if the objects $N := E \otimes X$ and E are cellular, then it is satisfied precisely when N is a retract of a direct sum of suspensions of copies of E via E-module homomorphisms:

Proposition 5.14. *Let (E, μ, e) be a monoid object and (N, κ) an E-module object in $\mathcal{SH}$. If there exists a collection of $a_i \in A$ such that N is a retract of $\bigoplus_i (E \otimes S^{a_i})$ in E-**Mod**,*[3] *then $\pi_*(N)$ is a graded projective (Definition B.15) $\pi_*(E)$-module. If E and N are cellular, then the converse holds as well.*

Proof. First suppose that there exists some collection of $a_i \in A$ such that N is a retract of $M := \bigoplus_i (E \otimes S^{a_i})$ in E-**Mod**:

$$\begin{tikzcd} N \arrow[r] \arrow[rr, bend left, equal] & M \arrow[r] & N \end{tikzcd}$$

Then applying $\pi_*(-)$ to the above diagram yields the following diagram of A-graded left $\pi_*(E)$-modules:

$$\begin{tikzcd} \pi_*(N) \arrow[r] \arrow[rr, bend left, equal] & \pi_*(M) \arrow[r] & \pi_*(N) \end{tikzcd}$$

Note we have an isomorphism of A-graded left $\pi_*(E)$-modules $\pi_*(M) \xrightarrow{\cong} \bigoplus_i \pi_{*-a_i}(E)$ given by the composition

$$\pi_*(\bigoplus_i (E \otimes S^{a_i})) \xrightarrow{\cong} \bigoplus_i E_*(S^{a_i}) = \bigoplus_i E_*(\Sigma^{a_i} S) \xrightarrow{\bigoplus_i t_S^{a_i}} \bigoplus_i E_{*-a_i}(S) = \bigoplus_i \pi_{*-a_i}(E),$$

where the first isomorphism is because $\pi_* : E\text{-}\mathbf{Mod} \to \pi_*(E)\text{-}\mathbf{Mod}^A$ preserves arbitrary coproducts (by Proposition 5.7), the maps $t_S^{a_i} : E_*(\Sigma^{a_i} S) \xrightarrow{\cong} E_{*-a_i}(S)$ are the degree-preserving isomorphisms of A-graded left $\pi_*(E)$-modules from Lemma 6.10, and the equalities follow from the coherence theorem for monoidal categories, which tells us we may assume $S \otimes - = - \otimes S = \mathrm{Id}_{\mathcal{SH}}$. Hence $\pi_*(N)$ is isomorphic in $\pi_*(E)$-$\mathbf{Mod}^A$ to a free $\pi_*(E)$ module, so that $\pi_*(N)$ is a retract in $\pi_*(E)$-$\mathbf{Mod}^A$ of a free $\pi_*(E)$-module, meaning it is graded projective, as desired.

On the other hand, suppose that E and N are cellular and $\pi_*(N)$ is graded projective, and pick some homogeneous generating set $\{x_i\} \subseteq \pi_*(N)$. Let $M := \bigoplus_i (E \otimes S^{|x_i|})$. We have a map

$$r : M \to N$$

induced by the maps

$$r_i : E \otimes S^{|x_i|} \xrightarrow{E \otimes x_i} E \otimes N \xrightarrow{\kappa} N.$$

[3] Here $\bigoplus_i (E \otimes S^{a_i})$ is a coproduct (Proposition C.14) of a bunch of free E-module objects (Proposition C.12), so it is itself an E-module object.

This is a homomorphism of E-module objects:

$$\begin{tikzcd}
E\otimes\bigoplus_i(E\otimes S^{|x_i|}) \arrow[rr, "E\otimes r"] \arrow[dr, "E\otimes\bigoplus_i r_i"] \arrow[dd, "\cong"'] & & E\otimes N \arrow[dddd, "\kappa"] \\
& E\otimes\bigoplus_i N \arrow[ur, "E\otimes\nabla"] \arrow[d, "\cong"'] & \\
\bigoplus_i(E\otimes E\otimes S^{|x_i|}) \arrow[r, "\bigoplus_i(E\otimes r_i)"] \arrow[dd, "\bigoplus_i(\mu\otimes S^{|x_i|})"'] & \bigoplus_i(E\otimes N) \arrow[uur, "\nabla"'] \arrow[d, "\bigoplus_i\kappa"] & \\
& \bigoplus_i N \arrow[dr, "\nabla"] & \\
\bigoplus_i(E\otimes S^{|x_i|}) \arrow[ur, "\bigoplus_i r_i"] \arrow[rr, "r"] & & N
\end{tikzcd}$$

The right trapezoid commutes by naturality of ∇. The bottom triangle commutes by the fact that $\nabla\circ\bigoplus_i r_i$ and r satisfy the same universal property for the coproduct. Every other region commutes by additivity of $E\otimes -$, except the left trapezoid: Note that by expanding out how r_i is defined, it becomes

$$\begin{tikzcd}
\bigoplus_i(E\otimes E\otimes S^{|x_i|}) \arrow[r, "\bigoplus_i(E\otimes E\otimes x_i)"] \arrow[d, "\bigoplus_i(\mu\otimes S^{|x_i|})"'] & \bigoplus_i(E\otimes E\otimes N) \arrow[r, "\bigoplus_i(E\otimes\kappa)"] \arrow[d, "\bigoplus_i(\mu\otimes X)"] & \bigoplus_i(E\otimes E\otimes X) \arrow[d, "\bigoplus_i\kappa"] \\
\bigoplus_i(E\otimes S^{|x_i|}) \arrow[r, "\bigoplus_i(E\otimes x_i)"'] & \bigoplus_i(E\otimes N) \arrow[r, "\bigoplus_i\kappa"'] & \bigoplus_i(E\otimes X)
\end{tikzcd}$$

The left square commutes by functoriality of $-\otimes-$, and the right square commutes by coherence for κ. Hence, we've shown that r is a homomorphism of E-modules, as desired. Thus, r induces a homomorphism of left $\pi_*(E)$-modules $\pi_*(r)\in\mathrm{Hom}_{\pi_*(E)}(\pi_*(M),\pi_*(N))$. Further note that for all $i\in I$, x_i is in the image of $\pi_*(r)$, as by definition $\pi_*(r)$ sends the class

$$S^{|x_i|}\xrightarrow{e\otimes S^{|x_i|}}E\otimes S^{|x_i|}\hookrightarrow M$$

in $\pi_{|x_i|}(M)$ to the composition

$$S^{|x_i|}\xrightarrow{e\otimes S^{|x_i|}}E\otimes S^{|x_i|}\xrightarrow{E\otimes x_i}E\otimes N\xrightarrow{\kappa}N,$$

and by unitality of κ this composition is simply $x_i : S^{|x_i|}\to N$. Thus, we have constructed a surjective A-graded homomorphism $\pi_*(r):\pi_*(M)\to\pi_*(N)$ of left $\pi_*(E)$-modules, so that since $\pi_*(N)$ is projective graded module there exists an A-graded left $\pi_*(E)$-module homomorphism $\iota:\pi_*(N)\to\pi_*(M)$ which makes the following diagram commute:

$$\begin{tikzcd}
& \pi_*(M) \arrow[d, "\pi_*(r)"] \\
\pi_*(N) \arrow[ur, "\iota"] \arrow[r, equal] & \pi_*(N)
\end{tikzcd}$$

Thus we have an idempotent of left A-graded $\pi_*(E)$-modules:

$$\pi_*(M)\xrightarrow{\pi_*(r)}\pi_*(N)\xrightarrow{\iota}\pi_*(M)$$

Now, by Proposition 5.15, since $M=\bigoplus_i(E\otimes S^{|x_i|})$, we have that the map

$$\pi_*:\mathrm{Hom}_{E\text{-}\mathbf{Mod}}(M,M)\to\mathrm{Hom}_{\pi_*(E)\text{-}\mathbf{Mod}}(\pi_*(M),\pi_*(M))$$

is an isomorphism of abelian groups, so that the above idempotent is induced by some endomorphism $\ell : M \to M$ of E-module objects. Further note that by functoriality of π_*,

$$\pi_*(\ell \circ \ell) = \pi_*(\ell) \circ \pi_*(\ell) = \pi_*(\ell),$$

and again since π_* is an isomorphism here, we have that $\ell \circ \ell = \ell$, so that ℓ is an idempotent in $\mathcal{SH}$. By Lemma 3.7, every idempotent in $\mathcal{SH}$ splits, meaning ℓ factors in $\mathcal{SH}$ as

$$\ell : M \xrightarrow{r'} X \xrightarrow{\iota'} M$$

with $r' \circ \iota' = \mathrm{id}_X$. Since X is a retract of an E-module object, and the corresponding idempotent is an E-module homomorphism, it follows purely formally that X may be canonically viewed as an E-module object, and that $r' : M \to X$ and $\iota' : X \to M$ are homomorphisms of E-module objects (see Lemma C.13 for details). Note that since E and each $S^{|x_i|}$ are cellular, $E \otimes S^{|x_i|}$ is cellular for all $i \in I$ (by Lemma 3.4), so that $M = \bigoplus_i (E \otimes S^{|x_i|})$ is cellular, as by definition an arbitrary coproduct of cellular objects is cellular. Thus since X is a retract of a cellular object in a triangulated category with arbitrary coproducts, it follows that X is cellular as well (see Lemma 3.7 for details). Now consider the following commutative diagram

$$\begin{tikzcd}
 & & \pi_*(N) \arrow[rr, equal] \arrow[dr, "\iota"] & & \pi_*(N) \arrow[dr, "\iota"] & & \\
\pi_*(N) \arrow[r, "\iota"'] \arrow[urr, equal, bend left] & \pi_*(M) \arrow[ur, "\pi_*(r)"] \arrow[rr, "\pi_*(\ell)"] \arrow[dr, "\pi_*(r')"'] & & \pi_*(M) \arrow[ur, "\pi_*(r)"] \arrow[rr, "\pi_*(\ell)"] \arrow[dr, "\pi_*(r')"] & & \pi_*(M) \arrow[r, "\pi_*(r')"] & \pi_*(X) \\
 & & \pi_*(X) \arrow[rr, equal] \arrow[ur, "\pi_*(\iota')"'] & & \pi_*(X) \arrow[ur, "\pi_*(\iota')"'] \arrow[urr, equal, bend right] & &
\end{tikzcd}$$

From this diagram we read off that the middle diagonal composition

$$\pi_*(X) \xrightarrow{\pi_*(\iota')} \pi_*(M) \xrightarrow{\pi_*(r)} \pi_*(N)$$

is an isomorphism with inverse $\pi_*(r') \circ \iota$. Now, since X and N are cellular, and $\pi_*(r \circ \iota')$ is an isomorphism, by Lemma 3.2 we have that $r \circ \iota'$ is an isomorphism, say with inverse p. Thus we have a commuting diagram

$$\begin{tikzcd}
N \arrow[rr, "\iota' \circ p"'] \arrow[dr, "p"'] \arrow[rrr, equal, bend left] & & M \arrow[r, "r"'] & N \\
 & X \arrow[ur, "\iota'"'] & &
\end{tikzcd}$$

and the middle row exhibits N as a retract of $M = \bigoplus_i (E \otimes S^{|x_i|})$, as desired. It remains to show this is a retract in E-**Mod**, i.e., that r and $\iota' \circ p$ are homomorphisms of E-module objects. Above we constructed r to be a homomorphism of E-modules. We also know that X is an E-module object and that ι' is an E-module homomorphism. Thus, it remains to show that $p : N \to X$ is an E-module homomorphism. But we know that p is the inverse of $r \circ \iota'$ in $\mathcal{SH}$, and we know r and ι' are morphisms in E-**Mod**, so that p is the inverse of $r \circ \iota'$ in E-**Mod**, meaning p is indeed an E-module homomorphism as desired. □

It turns out that the condition that N is a retract of a direct sum of suspensions of E is really the key condition which allows the universal coefficient theorem to go through, as the following two propositions illustrate.

Proposition 5.15. *Let (E, μ, e) be a monoid object and (N, κ) an E-module object in $\mathcal{SH}$. Then given a collection of $a_i \in A$ indexed by some set I, if (N, κ) is a retract of $\bigoplus_i (E \otimes S^{a_i})$ in*

E-$\mathbf{Mod}$,[4] *then for all E-module objects (N', κ'), the functor $\pi_* : E\text{-}\mathbf{Mod} \to \pi_*(E)\text{-}\mathbf{Mod}(A)$ (Proposition 5.7) induces an isomorphism of abelian groups*

$$\pi_* : \mathrm{Hom}_{E\text{-}\mathbf{Mod}}(N, N') \to \mathrm{Hom}_{\pi_*(E)}(\pi_*(N), \pi_*(N')).$$

Proof. To start, we consider the case $N = \bigoplus_i (E \otimes S^{a_i})$. Consider the following diagram:

$$\begin{tikzcd}
\mathrm{Hom}_E(\bigoplus_i (E \otimes S^{a_i}), N') \arrow[r, "\pi_*"] \arrow[d, "\cong"'] & \mathrm{Hom}_{\pi_*(E)}(\pi_*(\bigoplus_i (E \otimes S^{a_i})), \pi_*(N')) \arrow[d, "\cong"] \\
\prod_i \mathrm{Hom}_E(E \otimes S^{a_i}, N') \arrow[d, "\cong"'] & \mathrm{Hom}_{\pi_*(E)}(\bigoplus_i \pi_*(E \otimes S^{a_i}), \pi_*(N')) \arrow[d, "\cong"] \\
\prod_i [S^{a_i}, N'] \arrow[dd, equal] & \prod_i \mathrm{Hom}_{\pi_*(E)}(\pi_*(E \otimes S^{a_i}), \pi_*(N')) \arrow[d, "\cong"] \\
 & \prod_i \mathrm{Hom}_{\pi_*(E)}(\pi_{*-a_i}(E), \pi_*(N')) \arrow[d, equal] \\
\prod_i \pi_{a_i}(N') & \prod_i \mathrm{Hom}^{a_i}_{\pi_*(E)}(\pi_*(E), \pi_*(N')) \arrow[l, "\prod_i \mathrm{ev}_1"']
\end{tikzcd}$$

Here the top left vertical isomorphism exibits the universal property of the coproduct in E-$\mathbf{Mod}$, and middle left vertical isomorphism below that is the free-forgetful adjunction for E-modules (Proposition C.12). The bottom horizontal isomorphism is the product of the evaluation-at-1 isomorphisms (Lemma B.10). On the other side, the top right vertical isomorphism is given by the fact that $\pi_* : E\text{-}\mathbf{Mod} \to \pi_*(E)\text{-}\mathbf{Mod}^A$ preserves arbitrary coproducts (by Proposition 5.7). The middle right vertical isomorphism exhibits the universal property of the coproduct of modules. Finally the bottom right vertical isomorphism is given by the composition

$$\pi_*(E \otimes S^{a_i}) = E_*(S^{a_i}) = E_*(\Sigma^{a_i} S) \xrightarrow{t_S^{a_i}} E_{*-a_i}(S) = \pi_{*-a_i}(E),$$

where $t_X^{a_i}$ is the A-graded isomorphism of left $\pi_*(E)$-modules from Lemma 6.10, and the second and last equalities follow by the coherence theorem for monoidal categories, which tells us we may assume $S \otimes -$ and $- \otimes S$ are the identity on $\mathcal{SH}$. Now, we claim this diagram commutes. This really simply amounts to unravelling definitions, and chasing a homomorphism $f : \bigoplus_i (E \otimes S^{a_i}) \to N'$ of E-module objects both ways around the diagram yields the composition

$$\prod_i (S^{a_i} \xrightarrow{e \otimes S^{a_i}} E \otimes S^{a_i} \hookrightarrow \bigoplus_i (E \otimes S^{a_i}) \xrightarrow{f} N').$$

Thus, since the diagram commutes, we have that

$$\pi_* : \mathrm{Hom}_E(\bigoplus_i (E \otimes S^{a_i}), N') \to \mathrm{Hom}_{\pi_*(E)}(\pi_*(\bigoplus_i (E \otimes S^{a_i})), \pi_*(N'))$$

is an isomorphism, as desired.

Now, consider the case that N is a retract of $\bigoplus_i (E \otimes S^{a_i})$ in E-$\mathbf{Mod}$, so there exists a commuting diagram of E-module object homomorphisms:

$$\begin{tikzcd}
N \arrow[r, "\iota"'] \arrow[rr, bend left, equal] & \bigoplus_i (E \otimes S^{a_i}) \arrow[r, "r"'] & N
\end{tikzcd}$$

[4]Here $\bigoplus_i (E \otimes S^{a_i})$ is a coproduct (Proposition C.14) of a bunch of free E-module objects (Proposition C.12), so it is itself an E-module object.

Now consider the following diagram:

$$\begin{tikzcd}
\mathrm{Hom}_E(N,N') \arrow[r, "r^*"] \arrow[d, "\pi_*"'] \arrow[rr, bend left, equal] & \mathrm{Hom}_E(\bigoplus_i(E\otimes S^{a_i}),N') \arrow[r, "\iota^*"] \arrow[d, "\pi_*"] & \mathrm{Hom}_E(N,N') \arrow[d, "\pi_*"'] \\
\mathrm{Hom}_{\pi_*(E)}(\pi_*(N),\pi_*(N')) \arrow[r, "(\pi_*(r))^*"] \arrow[rr, bend right, equal] & \mathrm{Hom}_{\pi_*(E)}(\pi_*(\bigoplus_i(E\otimes S^{a_i})),\pi_*(N')) \arrow[r, "(\pi_*(\iota))^*"] & \mathrm{Hom}_{\pi_*(E)}(\pi_*(N),\pi_*(N'))
\end{tikzcd}$$

Each square commutes by functoriality of π_*. We have shown the middle vertical arrow is an isomorphism. Thus the outside arrows are isomorphisms as well, as a retract of an isomorphism is an isomorphism. ☐

Proposition 5.16. *Let (E,μ,e) be a monoid object and X an object in $\mathcal{SH}$. If there is a collection of $a_i \in A$ indexed by some set I such that $E\otimes X$ is a retract of $\bigoplus_i(E\otimes S^{a_i})$ in E-**Mod**,[5] then for all E-module objects (N,κ), the assignment*

$$[X,N]\to \mathrm{Hom}_{\pi_*(E)}(E_*(X),\pi_*(N)), \qquad [X\xrightarrow{f}N]\mapsto[\pi_*(\kappa)\circ E_*(f)]$$

is an isomorphism, and further extends to an A-graded isomorphism of A-graded abelian groups

$$[X,N]_*\to \mathrm{Hom}^*_{\pi_*(E)}(E_*(X),\pi_*(N)).$$

Proof. For each $a\in A$, define

$$U_a : [X,N]_a \to \mathrm{Hom}^a_{\pi_*(E)}(E_*(X),\pi_*(N))$$

to be the composition

$$\begin{tikzcd}
{[X,N]_a} \arrow[r, equal] & {[\Sigma^a X,N]} \arrow[d, "\mathrm{adj}"] & \\
& \mathrm{Hom}_{E\text{-}\mathbf{Mod}}(E\otimes\Sigma^a X,N) \arrow[d, "\pi_*(-)"] & \\
& \mathrm{Hom}_{\pi_*(E)}(E_*(\Sigma^a X),\pi_*(N) \arrow[d, "((t^a_X)^{-1})^*"] & \\
& \mathrm{Hom}_{\pi_*(E)}(E_{*-a}(X),\pi_*(N)) \arrow[r, equal] & \mathrm{Hom}^a_{\pi_*(E)}(E_*(X),\pi_*(N))
\end{tikzcd}$$

where the first isomorphism is the free-forgetful adjunction for E-modules (Proposition C.12), the second map is that induced by the functor π_* constructed in Proposition 5.7, and the third map is induced by the A-graded isomorphism of left $\pi_*(E)$-modules $(t^a_X)^{-1} : E_{*-a}(X)\to E_*(\Sigma^a X)$ from Lemma 4.4. By unravelling definitions, is straightforward to see that under the identification $[X,N]\cong[X,N]_0$, the map $U_0 : [X,N]_0 \to \mathrm{Hom}^0_{\pi_*(E)}(E_*(X),\pi_*(N))$ coincides with the assignment

$$[X,N]\to \mathrm{Hom}_{\pi_*(E)}(E_*(X),\pi_*(N)) \qquad [X\xrightarrow{f}N]\mapsto[\pi_*(\kappa)\circ\pi_*(E\otimes f)].$$

Furthermore, note we have isomorphisms in E-**Mod**

$$E\otimes\Sigma^a X = E\otimes S^a\otimes X \cong S^a\otimes E\otimes X$$

[5]Here $\bigoplus_i(E\otimes S^{a_i})$ is a coproduct (Proposition C.14) of a bunch of free E-module objects (Proposition C.12), so it is itself a E-module object.

(by Lemma 5.11) and

$$S^a \otimes \bigoplus_i (E \otimes S^{a_i}) \cong \bigoplus_i (S^a \otimes E \otimes S^{a_i}) \cong \bigoplus_i (E \otimes S^a \otimes S^{a_i}) \cong \bigoplus_i (E \otimes S^{a+a_i}),$$

where the first isomorphism is in E-**Mod** by Lemma 5.12, the second is in E-**Mod** by Lemma 5.11, and the last is a coproduct of homomorphisms of free E-modules (Proposition C.12), so it is also an E-module homomorphism. Hence we have that $E \otimes \Sigma^a X \cong S^a \otimes E \otimes X$ is a retract of $\bigoplus_i (E \otimes S^{a+a_i}) \cong S^a \otimes \bigoplus_i (E \otimes S^{a_i})$ in E-**Mod**, as $E \otimes X$ is a retract of $\bigoplus_i (E \otimes S^{a_i})$ in E-**Mod**, so that by Proposition 5.15, the map

$$\pi_* : \mathrm{Hom}_{E\text{-}\mathbf{Mod}}(E \otimes \Sigma^a X, N) \to \mathrm{Hom}_{\pi_*(E)}(E_*(\Sigma^a X), \pi_*(N))$$

is an isomorphism. Thus, we have constructed a bunch of isomorphisms

$$U_a : [X, N]_a \to \mathrm{Hom}^a_{\pi_*(E)}(E_*(X), \pi_*(N)),$$

so that by the universal property of the coproduct of abelian groups, there is a unique A-graded isomorphism

$$[X, N]_* \to \mathrm{Hom}^*_{\pi_*(E)}(E_*(X), \pi_*(N))$$

extending these maps, as desired. □

6. The dual E-Steenrod algebra

In Section 4.1, we showed that given a monoid object (E, μ, e) in $\mathcal{SH}$, that $E_*(E) = \pi_*(E \otimes E)$ is both a ring (since $E \otimes E$ is a monoid object if E is), and an A-graded bimodule over the ring $\pi_*(E)$. In this subsection, we will outline some additional structure carried by the pair $(E_*(E), \pi_*(E))$. Namely, we will show that if (E, μ, e) is a flat (Definition 6.5) commutative monoid object, then this pair, called the *dual E-Steenrod algebra*, is canonically an *A-graded anticommutative Hopf algebroid* over the stable homotopy ring $\pi_*(S)$ (Definition E.2). To start with, we outline some structure maps relating $E_*(E)$ and $\pi_*(E)$.

First, recall that given a monoid object (E, μ, e) in $\mathcal{SH}$, $\pi_*(E)$ is canonically an A-graded ring by Proposition 4.1, and so is $E_*(E) = \pi_*(E \otimes E)$ and $E_*(E \otimes E) = \pi_*(E \otimes E \otimes E)$, since the tensor product of monoid objects in a symmetric monoidal category is again a monoid object (Lemma C.4).

Lemma 6.1. *Let (E, μ, e) be a commutative monoid object in $\mathcal{SH}$. Then the maps*

(1) $E \xrightarrow{\cong} E \otimes S \xrightarrow{E \otimes e} E \otimes E$,
(2) $E \xrightarrow{\cong} S \otimes E \xrightarrow{e \otimes E} E \otimes E$,
(3) $E \otimes E \xrightarrow{\cong} E \otimes S \otimes E \xrightarrow{E \otimes e \otimes E} E \otimes E \otimes E$,
(4) $E \otimes E \xrightarrow{\mu} E$, *and*
(5) $E \otimes E \xrightarrow{\tau_{E,E}} E \otimes E$

are homomorphisms of monoid objects in $\mathcal{SH}$ (where here $E \otimes E$ and $E \otimes E \otimes E$ are considered as monoid objects in $\mathcal{SH}$ by Lemma C.4 and Lemma C.5, respectively), so that by Proposition 4.15, under π_* they induce morphisms in $\pi_*(S)$-$\mathbf{GCA}^A$:

(1) $\eta_L : \pi_*(E) \to E_*(E)$,
(2) $\eta_R : \pi_*(E) \to E_*(E)$,
(3) $h : E_*(E) \to E_*(E \otimes E)$,
(4) $\epsilon : E_*(E) \to \pi_*(E)$, *and*
(5) $c : E_*(E) \to E_*(E)$.

Proof. It is a general fact that the unit and multiplication maps $e : S \to E$ and $\mu : E \otimes E \to E$ for a monoid are monoid homomorphisms when (E, μ, e) is a commutative monoid object (Lemma C.6), so that the maps $E \otimes e$, and $e \otimes E$ from E to $E \otimes E$ are monoid homomorphisms, by Lemma C.7.

Similarly, $E \otimes e \otimes E : E \otimes E \to E \otimes E \otimes E$ is a monoid homomorphism. Thus, it remains to show that $\tau_{E,E} : E \otimes E \to E \otimes E$ is a monoid homomorphism. First, consider the following diagram:

$$\begin{tikzcd}
E_1 \otimes E_2 \otimes E_3 \otimes E_4 \arrow[r, "\tau\otimes\tau"] \arrow[d, "E\otimes\tau\otimes E"'] & E_2 \otimes E_1 \otimes E_4 \otimes E_3 \arrow[d, "E\otimes\tau\otimes E"] \\
E_1 \otimes E_3 \otimes E_2 \otimes E_4 \arrow[r, "\tau_{E\otimes E, E\otimes E}"] \arrow[d, "\mu\otimes\mu"'] & E_2 \otimes E_4 \otimes E_1 \otimes E_3 \arrow[d, "\mu\otimes\mu"] \\
E_{1,3} \otimes E_{2,4} \arrow[r, "\tau"'] & E_{2,4} \otimes E_{1,3}
\end{tikzcd}$$

(Here we've labelled the E's to make the action of the braidings clearer). The top region commutes by coherence for the symmetries in a symmetric monoidal category, while the bottom region commutes by naturality of τ. Now, consider the following diagram:

$$\begin{tikzcd}
 & & S \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & & \\
 & S \otimes S \arrow[rr, "\tau"] \arrow[dl, "e\otimes e"'] & & S \otimes S \arrow[dr, "e\otimes e"] & \\
E \otimes E \arrow[rrrr, "\tau"] & & & & E \otimes E
\end{tikzcd}$$

The top triangle commutes by coherence for a symmetric monoidal category, while the bottom region commutes by naturality of τ. Thus, we have shown $\tau_{E,E}$ is a homomorphism of monoid objects, as desired. $\square$

Recall a that given a homomorphism of rings $f : R \to R'$, the ring R' canonically becomes an R-bimodule with left action $r \cdot x := f(r)x$ and right action $x \cdot r := xf(r)$. In particular, the ring homomorphisms $\eta_L : \pi_*(E) \to E_*(E)$ and $\eta_R : \pi_*(E) \to E_*(E)$ endow $E_*(E)$ with the structure of a bimodule over $\pi_*(E)$. Naturally, one may ask in what sense these bimodule structures coincide with the canonical one (from Proposition 4.3). The following lemma tells us that the canonical $\pi_*(E)$-bimodule structure on $E_*(E)$ is that with left action induced by η_L and right action induced by η_R:

Lemma 6.2. *Let (E, μ, e) be a commutative monoid object in $\mathcal{SH}$. Then the left (resp. right) $\pi_*(E)$-module structure induced on $E_*(E)$ by the ring homomorphism η_L (resp. η_R) coincides with the canonical left (resp. right) $\pi_*(E)$-module structure on $E_*(E)$ given in Proposition 4.3.*

Proof. What's going on here is a bit subtle, so we're going to be really explicit. In Proposition 4.3, it was shown that $E_*(E)$ is a left $\pi_*(E)$-module via the assignment

$$\pi_*(E) \times E_*(E) \to E_*(E)$$

which sends homogeneous elements $r : S^a \to E$ and $x : S^b \to E \otimes E$ to the composition

$$S^{a+b} \xrightarrow{\cong} S^a \otimes S^b \xrightarrow{r\otimes x} E \otimes E \otimes E \xrightarrow{\mu\otimes E} E \otimes E.$$

We'd like to show that this is the same thing as the assignment $\pi_*(E) \times E_*(E) \to E_*(E)$ sending $(r, x) \mapsto \eta_L(r)x$, where $\eta_L(r)x$ denotes the product of $\eta_L(r)$ and x taken in the ring $E_*(E)$. Explicitly, the product structure on $E_*(E) = \pi_*(E \otimes E)$ is that induced by the fact that $E \otimes E$ is a monoid object in $\mathcal{SH}$ (by Lemma C.4), with product

$$E \otimes E \otimes E \otimes E \xrightarrow{E\otimes\tau\otimes E} E \otimes E \otimes E \otimes E \xrightarrow{\mu\otimes\mu} E \otimes E$$

(note the middle two factors are swapped). By linearity of module actions, in order to show the canonical left $\pi_*(E)$-module structure on $E_*(E)$ agrees with that induced by η_L, it suffices to show the module actions agree on homogeneous elements. Now, suppose we have homogeneous elements $r : S^a \to E$ in $\pi_*(E)$ and $x : S^b \to E \otimes E$ in $E_*(E)$, and consider the following diagram, where we've passed to a symmetric strict monoidal category:

$$\begin{tikzcd}
S^{a+b} \arrow[d, "\phi_{a,b}"] & & & & \\
S^a \otimes S^b \arrow[d, "r\otimes x"] & & & & \\
E_1 \otimes E_2 \otimes E_3 \arrow[rrrr, "\mu\otimes E"] \arrow[dd, "E\otimes e\otimes E"'] \arrow[dr, equal] & & & & E_{1,2} \otimes E_3 \arrow[dd, equal] \\
 & E_1 \otimes E_2 \otimes E_3 \arrow[r, equal] & E_1 \otimes E_2 \otimes E_3 \arrow[r, equal] \arrow[d, "E\otimes E\otimes e\otimes E"] & E_1 \otimes E_2 \otimes E_3 \arrow[dr, "\mu\otimes E"] & \\
E_1 \otimes E \otimes E_2 \otimes E_3 \arrow[ur, "E\otimes\mu\otimes E"] \arrow[rr, "E\otimes\tau\otimes E"'] & & E_1 \otimes E_2 \otimes E \otimes E_3 \arrow[ul, "E\otimes\mu\otimes E"] \arrow[ur, "E\otimes E\otimes\mu"'] \arrow[rr, "\mu\otimes\mu"'] & & E_{1,2} \otimes E_3
\end{tikzcd}$$

Here we've numbered the E's to make it clear what's going on. The bottom composition is $\eta_L(r)x$, while the top composition is the canonical left action of r on x given in Proposition 4.3. The leftmost triangle commutes by unitality of μ. The triangle to the right of that commutes by commutativity of μ. The triangle to the right of that commutes by unitality of μ, as does the next triangle. The remaining triangle on the right commutes by functoriality of $- \otimes -$. Finally, the top region commutes by definition. Thus, we've shown that the left $\pi_*(E)$-module structure induced on $E_*(E)$ by η_L is in fact the canonical one. On the other hand, showing that the right $\pi_*(E)$-module structure induced on $E_*(E)$ by η_R is the canonical one is entirely analagous, and we leave it as an exercise for the reader. $\square$

Recall (Proposition B.22) that the pushout of two morphisms $f : B \to C$ and $g : B \to D$ in R-$\mathbf{GCA}^A$ is obtained by taking the tensor product of B-modules $C \otimes_B D$, where C has right B-module action induced by f, and D has left B-module action induced by g, and giving it an anticommutative product which makes $C \otimes_B D$ a ring. Thus, by the above lemma, we may view the tensor product of bimodules $E_*(E) \otimes_{\pi_*(E)} E_*(E)$ (where $E_*(E)$ is considered with its canonical $\pi_*(E)$-bimodule structure from Proposition 4.3) as not just an A-graded abelian group or a $\pi_*(E)$-bimodule, but as an A-graded anticommutative $\pi_*(S)$-algebra:

Corollary 6.3. *Given a commutative monoid object (E, μ, e) in $\mathcal{SH}$, the domain of the homomorphism*

$$\Phi_{E,E} : E_*(E) \otimes_{\pi_*(E)} E_*(E) \to E_*(E \otimes E)$$

constructed in Proposition 5.6 is canonically an A-graded anticommutative $\pi_*(S)$-algebra, and sits in the following pushout diagram in $\pi_*(S)$-$\mathbf{GCA}^A$:

$$\begin{tikzcd}
\pi_*(E) \arrow[r, "\eta_L"] \arrow[d, "\eta_R"'] & E_*(E) \arrow[d, "x\mapsto 1\otimes x"] \\
E_*(E) \arrow[r, "x\mapsto x\otimes 1"'] & E_*(E) \otimes_{\pi_*(E)} E_*(E)
\end{tikzcd}$$

Furthermore, with respect to this ring structure, $\Phi_{E,E}$ is a homomorphism of rings:

Lemma 6.4. *Let (E, μ, e) be a commutative monoid object in $\mathcal{SH}$. Then the homomorphism*

$$\Phi_{E,E} : E_*(E) \otimes_{\pi_*(E)} E_*(E) \to E_*(E \otimes E)$$

constructed in Proposition 5.2 is a homomorphism of A-graded anticommutative $\pi_*(S)$-algebras.

Proof. Consider the maps

$$f : E \otimes E \xrightarrow{e \otimes E \otimes E} E \otimes E \otimes E$$

and

$$g : E \otimes E \xrightarrow{E \otimes E \otimes e} E \otimes E \otimes E.$$

We know that the maps

$$E \xrightarrow{e \otimes E} E \otimes E \qquad \text{and} \qquad E \xrightarrow{E \otimes e} E \otimes E$$

are monoid homomorphisms by Lemma 6.1, so that f and g are monoid homomorphisms by Lemma C.7. Furthermore, by Lemma C.5, they are monoid homomorphisms between the same monoid objects in $\mathcal{SH}$ (when we assume that strict associativity holds). Finally, note that we have the following commutative diagram

$$\begin{tikzcd}
E \arrow[r, "E\otimes e"] \arrow[d, "e\otimes E"'] \arrow[dr, "e\otimes E\otimes e" description] & E \otimes E \arrow[d, "e\otimes E\otimes E"] \\
E \otimes E \arrow[r, "E\otimes E\otimes e"'] & E \otimes E \otimes E
\end{tikzcd}$$

where the outer arrows are monoid object homomorphisms, thus, we may apply π_*, which yields the following commutative diagram in $\pi_*(S)$-$\mathbf{GCA}^A$ (Proposition 4.15):

$$\begin{tikzcd}
\pi_*(E) \arrow[r, "\eta_L"] \arrow[d, "\eta_R"'] & E_*(E) \arrow[d, "\pi_*(f)"] \\
E_*(E) \arrow[r, "\pi_*(g)"'] & E_*(E \otimes E)
\end{tikzcd}$$

Hence by Lemma 6.4 and the universal property of the pushout, there exists some unique morphism $\ell : E_*(E) \otimes_{\pi_*(E)} E_*(E) \to E_*(E \otimes E)$ in $\pi_*(S)$-$\mathbf{GCA}^A$ which makes the following diagram commute:

$$\begin{tikzcd}
\pi_*(E) \arrow[r, "\eta_L"] \arrow[d, "\eta_R"'] & E_*(E) \arrow[d, "x\mapsto 1\otimes x"'] \arrow[ddr, bend left, "\pi_*(f)"] & \\
E_*(E) \arrow[r, "x\mapsto x\otimes 1"] \arrow[drr, bend right, "\pi_*(g)"'] & E_*(E) \otimes_{\pi_*(E)} E_*(E) \arrow[dr, dashed, "\ell"] & \\
& & E_*(E \otimes E)
\end{tikzcd}$$

Thus in order to show $\Phi_{E,E}$ is a morphism in $\pi_*(S)$-$\mathbf{GCA}^A$, it suffices to show that $\Phi_{E,E}$ and ℓ are the same map, since we know ℓ is a homomorphism of A-graded anticommutative $\pi_*(S)$-algebras. Since $\Phi_{E,E}$ and ℓ are both abelian group homomorphisms, it further suffices to show they agree on homogeneous pure tensors, which generate $E_*(E) \otimes_{\pi_*(E)} E_*(E)$ as an abelian group. Given homogeneous elements $x : S^a \to E \otimes E$ and $y : S^b \to E \otimes E$ in $E_*(E)$, unravelling how pushouts in $\pi_*(S)$-$\mathbf{GCA}^A$ are defined (Proposition B.22), ℓ sends the pure homogeneous tensor $x \otimes y$ to the element $\pi_*(g)(x) \cdot \pi_*(f)(y)$, where here $\cdot$ denotes the product taken in $E_*(E \otimes E) = \pi_*(E \otimes E \otimes E)$.

Now, consider the following diagram:

$$
\begin{tikzcd}
S^{a+b} \arrow[d, "\phi_{a,b}"] & & \\
S^a \otimes S^b \arrow[d, "x\otimes y"] & & \\
E_1 \otimes E_2 \otimes E_3 \otimes E_4 \arrow[rr, "g\otimes f = E\otimes E\otimes e\otimes e\otimes E\otimes E"] \arrow[drr, "E\otimes e\otimes E\otimes e\otimes E\otimes E"] \arrow[ddr, equal] \arrow[ddd, "E\otimes\mu\otimes E"'] & & E_1 \otimes E_2 \otimes E_a \otimes E_b \otimes E_3 \otimes E_4 \arrow[d, "E\otimes\tau_{E\otimes E,E}\otimes E\otimes E"] \\
& & E_1 \otimes E_b \otimes E_2 \otimes E_a \otimes E_3 \otimes E_4 \arrow[d, "\mu\otimes E\otimes\tau\otimes E"] \\
& E_1 \otimes E_2 \otimes E_3 \otimes E_4 \arrow[ur, "E\otimes e\otimes E\otimes e\otimes E\otimes E"] \arrow[r, "E\otimes E\otimes E\otimes e\otimes E"'] \arrow[dl, "E\otimes\mu\otimes E"] & E_1 \otimes E_2 \otimes E_3 \otimes E_a \otimes E_4 \arrow[d, "E\otimes\mu\otimes\mu"] \\
E_1 \otimes E_{2,3} \otimes E_4 \arrow[rr, equal] & & E_1 \otimes E_{2,3} \otimes E_4
\end{tikzcd}
$$

Here we have labelled the E's to make things clearer. The left outside composition is $\Phi_{E,E}(x\otimes y)$, while the right composition is $\pi_*(g)(x) \cdot \pi_*(f)(y)$. The top right triangle commutes by coherence for a symmetric monoidal category. The middle tright triangle commutes by unitality of μ and coherence for a symmetric monoidal category. The bottom trapezoid commutes by unitality of μ. The rest of the diagram commutes by definition. Thus we have $\Phi_{E,E}(x\otimes y) = \pi_*(g)(x)\cdot\pi_*(f)(y)$, so that $\Phi_{E,E} = \ell$ is not just an isomorphism of left $\pi_*(E)$-modules, but an isomorphism of A-graded anticommutative $\pi_*(S)$-algebras, as desired. $\square$

For the sake of conciseness, we make the following definition:

Definition 6.5. We say that a monoid object (E, μ, e) in $\mathcal{SH}$ is *flat* if the canonical right $\pi_*(E)$-module structure on $E_*(E)$ from Proposition 4.3 is that of a flat module, or equivalently by Lemma 6.2, if the the map $\eta_R : \pi_*(E) \to E_*(E)$ constructed in Lemma 6.1 is a flat ring homomorphism.

Finally, we can package all of this information into an object called the *dual E-Steenrod algebra*:

Definition 6.6. Let (E, μ, e) be a *commutative* monoid object in $\mathcal{SH}$ which is flat (Definition 6.5) and cellular (Definition 3.1). Then the *dual E-Steenrod algebra* is the pair of A-graded abelian groups $(E_*(E), \pi_*(E))$ equipped with the following structure:

1. The A-graded anticommutative $\pi_*(S)$-algebra structure on $\pi_*(E)$ induced from E being a commutative monoid object in $\mathcal{SH}$ (Proposition 4.15).
2. The A-graded anticommutative $\pi_*(S)$-algebra structure on $E_*(E)$ induced from the fact that $E \otimes E$ is canonically a commutative monoid object in $\mathcal{SH}$ (Lemma C.4), so that also $E_*(E) = \pi_*(E \otimes E)$ is an A-graded anticommutative $\pi_*(S)$-algebra (Proposition 4.15).
3. The homomorphisms of A-graded anticommutative $\pi_*(S)$-algebras
$$\eta_L : \pi_*(E) \to E_*(E)$$
and
$$\eta_R : \pi_*(E) \to E_*(E)$$
induced under π_* by the monoid object homomorphisms
$$E \xrightarrow{\cong} E \otimes S \xrightarrow{E\otimes e} E \otimes E$$
and
$$E \xrightarrow{\cong} S \otimes E \xrightarrow{e\otimes E} E \otimes E.$$

4. The homomorphism of A-graded anticommutative $\pi_*(S)$-algebras
$$\Psi_E : E_*(E) \to E_*(E) \otimes_{\pi_*(E)} E_*(E)$$
given by the composition
$$E_*(E) \xrightarrow{h} E_*(E \otimes E) \xrightarrow{\Phi_{E,E}^{-1}} E_*(E) \otimes_{\pi_*(E)} E_*(E),$$
where h is a homomorphism of A-graded anticommutative $\pi_*(S)$-algebras induced under π_* by the monoid object homomorphism
$$E \otimes E \xrightarrow{\cong} E \otimes S \otimes E \xrightarrow{E \otimes e \otimes E} E \otimes E \otimes E,$$
and $\Phi_{E,E}$ is morphism constructed in Proposition 5.2, which is proven to be an isomorphism in Proposition 5.6 (since E is flat and cellular), and furthermore an isomorphism in $\pi_*(S)$-$\mathbf{GCA}^A$ by Lemma 6.4.
5. The homomorphism of A-graded anticommutative $\pi_*(S)$-algebras
$$\epsilon : E_*(E) \to \pi_*(E)$$
induced under π_* by the monoid object homomorphism
$$E \otimes E \xrightarrow{\mu} E.$$
6. The homomorphism of A-graded anticommutative $\pi_*(S)$-algebras
$$c : E_*(E) \to E_*(E)$$
induced under π_* from the monoid object homomorphism
$$E \otimes E \xrightarrow{\tau} E \otimes E.$$

The curious reader may wonder why we call $(E_*(E), \pi_*(E))$ the *dual* E-Steenrod algebra. The "dual" is there because the E-Steenrod algebra refers instead to the E-self cohomology $E^*(E) \cong [E, E]_{-*}$. Clasically, the Adams spectral sequence was originally constructed in such a way that the E_1 and E_2 pages could be characterized in terms of cohomology groups as modules over the E-Steenrod algebra, but it turns out that our approach using homology groups as comodules over the dual E-Steenrod algebra is somewhat better behaved in practice.

6.1. **The dual E-Steenrod algebra is a Hopf algebroid.** Above, given a flat and cellular commutative monoid object (E, μ, e) in $\mathcal{SH}$, we constructed an algebraic gadget $(E_*(E), \pi_*(E))$ in the category $\pi_*(S)$-$\mathbf{GCA}^A$ of A-graded anticommutative $\pi_*(S)$-algebras called the *dual E-Steenrod algebra*. In this subsection, we will show this object is an example of the general notion of an *A-graded anticommutative Hopf algebroid*:

Proposition 6.7. *Let (E, μ, e) be a commutative monoid object in $\mathcal{SH}$ which is flat (Definition 6.5) and cellular (Definition 3.1). Then the dual E-Steenrod algebra $(E_*(E), \pi_*(E))$ with the structure maps $(\eta_L, \eta_R, \Psi, \epsilon, c)$ from Definition 6.6 is an A-graded anticommutative Hopf algebroid over $\pi_*(S)$ (Definition E.2), i.e., a co-groupoid object in the category $\pi_*(S)$-$\mathbf{GCA}^A$.*

Proof. All that needs to be done is to show all the diagrams in Definition E.2 commute. This is nearly all entirely straightforward, the only real difficulty that arises is showing the co-associativity diagram holds. The argument is sketched in the case $\mathcal{SH}$ is the classical stable homotopy category in sufficent detail in Lecture 3 of the article [1] by Adams. The argument given there works essentially the exact same way here in our more general setting. $\square$

6.2. **Comodules over the dual E-Steenrod algebra.** Finally, we can identify some additional structure on E-homology groups of (cellular) objects in $\mathcal{SH}$ in terms of the Hopf algebroid structure on the dual E-Steenrod algebra.

Proposition 6.8. *Let (E, μ, e) be a flat (Definition 6.5) and cellular (Definition 3.1) commutative monoid object in $\mathcal{SH}$. Then $E_*(-)$ is an additive functor from the full subcategory $\mathcal{SH}$-**Cell** of cellular objects in $\mathcal{SH}$ to the category $E_*(E)$-$\mathbf{CoMod}^A$ of left A-graded comodules (Definition E.6) over the dual E-Steenrod algebra, which is an A-graded commutative Hopf algebroid over $\pi_*(S)$, by Proposition 6.7.*

*In particular, given an object X in $\mathcal{SH}$-**Cell**, we are viewing $E_*(X)$ with its canonical left $\pi_*(E)$-module structure (Proposition 4.3), and the action map is given by the composition*

$$\Psi_X : E_*(X) \xrightarrow{E_*(e\otimes X)} E_*(E\otimes X) \xrightarrow{\Phi_{E,X}^{-1}} E_*(E) \otimes_{\pi_*(E)} E_*(X).$$

Proof. Again, we refer the reader to Lecture 3 in [1], where this is shown in the classical stable homotopy category (although the proof carries over basically verbatim to our setting). □

Now, we can use this structure in order to identify the group of maps $X \to E \otimes Y$ with graded $E_*(E)$-comodule homomorphisms from $E_*(X)$ to $E_*(Y)$. First, we need the following two technical lemmas:

Lemma 6.9. *Let (E, μ, e) be a flat (Definition 6.5) and cellular (Definition 3.1) commutative monoid object in $\mathcal{SH}$. Then given an object X in $\mathcal{SH}$, the map*

$$\Phi_{E,X} : E_*(E) \otimes_{\pi_*(E)} E_*(X) \to E_*(E \otimes X)$$

constructed in Proposition 5.2 is a homomorphism of A-graded left Γ-comodules, where here by Proposition E.8 we are viewing $E_*(E) \otimes_{\pi_*(E)} E_*(X)$ as the co-free $E_*(E)$-comodule on $E_*(X)$ with its canonical A-graded left $\pi_*(E)$-module structure (from Proposition 4.3), and $E_*(E \otimes X)$ with its canonical left $E_*(E)$-comodule structure from Proposition 6.8.

Proof. Consider the following diagram:

$$\begin{tikzcd}
E_*(E) \otimes_{\pi_*(E)} E_*(X) \arrow[rr, "\Psi_{E_*(E)\otimes E_*(X)}"] \arrow[dddddd, "\Phi_{E,X}"'] \arrow[ddr, bend right, "E_*(e\otimes E)"'] & & E_*(E) \otimes_{\pi_*(E)} (E_*(E) \otimes_{\pi_*(E)} E_*(X)) \arrow[dddddd, "E_*(E)\otimes\Phi_{E,X}"] \\
& (E_*(E) \otimes_{\pi_*(E)} E_*(E)) \otimes_{\pi_*(E)} E_*(X) \arrow[ur, "\cong"'] \arrow[d, "\Phi_{E,E\otimes E_*(X)}"] & \\
& E_*(E \otimes E) \otimes_{\pi_*(E)} E_*(X) \arrow[d, equal] & \\
& (E \otimes E)_*(E) \otimes_{\pi_*(E)} E_*(X) \arrow[d, "\Phi_{E,X}"] & \\
& \pi_*(E \otimes E \otimes E \otimes X) \arrow[d, equal] & \\
& E_*(E \otimes E \otimes X) & \\
E_*(E \otimes X) \arrow[ur, "E_*(e\otimes E\otimes X)"] \arrow[rr, "\Psi_{E\otimes X}"] & & E_*(E) \otimes_{\pi_*(E)} E_*(E \otimes X) \arrow[ul, "\Phi_{E,E\otimes X}"']
\end{tikzcd}$$

The top and bottom regions commute by definition. The left region commutes by naturality of $\Phi_{E,X}$. Thus, it remains to show the rightmost region commutes. To that end, since all the arrows involved are homomorphisms, it suffices to chase a homogeneous pure tensor around. Let $x : S^a \to E \otimes E$, $y : S^b \to E \otimes E$, and $z : S^c \to E \otimes X$, and consider the following diagram:

$$\begin{tikzcd}
S^{a+b+c} \arrow[d, "\phi"'] & \\
S^a \otimes S^b \otimes S^c \arrow[d, "x\otimes y\otimes z"'] & \\
E \otimes E \otimes E \otimes E \otimes E \otimes X \arrow[r, "E\otimes\mu\otimes E\otimes E\otimes X"] \arrow[d, "E\otimes E\otimes E\otimes\mu\otimes X"'] & E \otimes E \otimes E \otimes E \otimes X \arrow[d, "E\otimes E\otimes\mu\otimes X"] \\
E \otimes E \otimes E \otimes E \otimes X \arrow[r, "E\otimes\mu\otimes E\otimes X"] & E \otimes E \otimes E \otimes X
\end{tikzcd}$$

The two compositions are the two results of chasing $(x \otimes y) \otimes z$ around the rightmost region in the above diagram. It clearly commutes by functoriality of $- \otimes -$. Hence, indeed we have that $\Phi_{E,X}$ is a homomorphism of left $E_*(E)$-comodules, as desired. $\square$

Lemma 6.10. *Let (E, μ, e) be a flat (Definition 6.5) and cellular (Definition 3.1) commutative monoid object in $\mathcal{SH}$. Then the isomorphism*

$$t^a_X : E_*(\Sigma^a X) \to E_{*-a}(X)$$

from Lemma 4.4 is an A-graded isomorphism of left $E_*(E)$-comodules.

Proof. We know that $t^a_X : E_*(\Sigma^a X) \to E_{*-a}(X)$ is already an A-graded isomorphism of left $\pi_*(E)$-modules, so clearly it simply suffices to show that t^a_X is a homomorphism of left $E_*(E)$-comodules. To that end, consider the following diagram:

$$\begin{tikzcd}
E_*(\Sigma^a X) \arrow[rr, "\Psi_{\Sigma^a X}"] \arrow[dr, "E_*(e\otimes\Sigma^a X)"'] \arrow[dddd, "t^a_X"'] & & E_*(E) \otimes_{\pi_*(E)} E_*(\Sigma^a X) \arrow[dl, "\Phi_{E,\Sigma^a X}"] \arrow[dddd, "E_*(E)\otimes t^a_X"] \\
& E_*(E \otimes \Sigma^a X) \arrow[d, dashed, "E_*(\tau_{E,S^a}\otimes X)"] & \\
& E_*(\Sigma^a(E \otimes X)) \arrow[d, dashed, "t^a_{E\otimes X}"] & \\
& E_{*-a}(E \otimes X) & \\
E_{*-a}(X) \arrow[rr, "\Psi_X"'] \arrow[ur, "E_{*-a}(e\otimes X)"] & & E_*(E) \otimes_{\pi_*(E)} E_{*-a}(X) \arrow[ul, "\Phi_{E,X}"']
\end{tikzcd} \tag{7}$$

The top and bottom regions commute by definition. To see the left and right regions commute, we'll do a diagram chase of homogeneous elements. First of all, let $x : S^b \to E \otimes S^a \otimes X$ in $E_*(\Sigma^a X)$, and consider the following diagram exhibiting the two ways to chase x around the leftmost region:

$$\begin{tikzcd}
S^{b-a} \arrow[d, "\phi_{b,-a}"'] & & & \\
S^b \otimes S^{-a} \arrow[d, "x \otimes S^{-a}"'] & & & \\
E \otimes S^a \otimes X \otimes S^{-a} \arrow[rr, "E \otimes e \otimes S^a \otimes X \otimes S^{-a}"] \arrow[dd, "E \otimes \tau \otimes S^{-a}"'] & & E \otimes E \otimes S^a \otimes X \otimes S^{-a} \arrow[r, "E \otimes \tau \otimes X \otimes S^{-a}"] \arrow[dr, "E \otimes E \otimes \tau \otimes S^{-a}"'] & E \otimes S^a \otimes E \otimes X \otimes S^{-a} \arrow[d, "E \otimes \tau_{S^a, E \otimes X} \otimes S^{-a}"] \\
& & & E \otimes E \otimes X \otimes S^a \otimes S^{-a} \arrow[d, "E \otimes E \otimes X \otimes \phi^{-1}_{a,-a}"] \\
E \otimes X \otimes S^a \otimes S^{-a} \arrow[r, "E \otimes X \otimes \phi^{-1}_{a,-a}"'] \arrow[urrr, "E \otimes e \otimes X \otimes S^a \otimes S^{-a}"] & E \otimes X \arrow[rr, "E \otimes e \otimes X"'] & & E \otimes E \otimes X
\end{tikzcd}$$

The top right region commutes by coherence for the symmetries, while the other two regions commute by functoriality of $- \otimes -$. Thus, it remains to show the rightmost region in diagram (7) commutes. To that end, let $x : S^b \to E \otimes E$ in $E_*(E)$ and $y : S^c \to E \otimes S^a \otimes X$ in $E_*(\Sigma^a X)$, and consider the following diagram, which exhibits the two ways to chase $x \otimes y$ around the rightmost region of diagram (7):

$$\begin{tikzcd}
S^{b+c-a} \arrow[d, "\phi"] & & \\
S^b \otimes S^c \otimes S^{-a} \arrow[d, "x \otimes y \otimes S^{-a}"] & & \\
E \otimes E \otimes E \otimes S^a \otimes X \otimes S^{-a} \arrow[r, "E \otimes \mu \otimes S^a \otimes X \otimes S^{-a}"] \arrow[d, "E \otimes E \otimes E \otimes \tau \otimes S^{-a}"'] & E \otimes E \otimes S^a \otimes X \otimes S^{-a} \arrow[r, "E \otimes \tau \otimes X \otimes S^{-a}"] \arrow[dr, "E \otimes E \otimes \tau \otimes S^{-a}"'] & E \otimes S^a \otimes E \otimes X \otimes S^{-a} \arrow[d, "E \otimes \tau_{S^a, E \otimes X} S^{-a}"] \\
E \otimes E \otimes E \otimes X \otimes S^a \otimes S^{-a} \arrow[rr, "E \otimes \mu \otimes X \otimes S^a \otimes S^{-a}"] \arrow[d, "E \otimes E \otimes E \otimes X \otimes \phi^{-1}_{a,-a}"'] & & E \otimes E \otimes X \otimes S^a \otimes S^{-a} \arrow[d, "E \otimes E \otimes X \otimes \phi^{-1}_{a,-a}"] \\
E \otimes E \otimes E \otimes X \arrow[rr, "E \otimes \mu \otimes X"] & & E \otimes E \otimes X
\end{tikzcd}$$

The top right region commutes by coherence for the symmetries. The remaining two regions commute by functoriality of $- \otimes -$. Thus, indeed we have that diagram (7) commutes, so t_X^a is a homomorphism of left $E_*(E)$-comodules, as desired. □

Now we may prove the theorem.

Theorem 6.11. *Let (E, μ, e) be a commutative monoid object, and X and Y objects in $\mathcal{SH}$. Suppose further that:*

- *E is flat (Definition 6.5) and cellular (Definition 3.1),*
- *X is cellular and $E_*(X)$ is a graded projective left $\pi_*(E)$-module (via Proposition 4.3), and*
- *Y is cellular.*

Then the assignment

$$E_*(-) : [X, E \otimes Y] \to \mathrm{Hom}_{E_*(E)}(E_*(X), E_*(E \otimes Y)), \qquad f \mapsto E_*(f)$$

induced by the functor $E_*(-) : \mathcal{SH}$-$\mathbf{Cell} \to E_*(E)$-$\mathbf{CoMod}^A$ (Proposition 6.8) is an isomorphism of abelian groups.

Proof. Since X is cellular, by Proposition 6.8 we have that $E_*(X)$ is canonically an A-graded left $E_*(E)$-comodule. Similarly, since E and Y are cellular, we know that $E \otimes Y$ is cellular, so that $E_*(E \otimes Y)$ is also canonically an $E_*(E)$-comodule. Thus, we have a well-defined assignment

$$[X, E \otimes Y] \xrightarrow{E_*(-)} \mathrm{Hom}_{E_*(E)}(E_*(X), E_*(E \otimes Y)).$$

To see this arrow is an isomorphism, consider the following diagram:

$$\begin{tikzcd}
{[X, E \otimes Y]} \arrow[rr, "E_*(-)"] \arrow[d, "\pi_*(\mu\otimes Y)\circ E_*(-)"'] & & \mathrm{Hom}_{E_*(E)}(E_*(X), E_*(E \otimes Y)) \arrow[dll, "\pi_*(\mu\otimes Y)\circ(-)"'] \\
\mathrm{Hom}_{\pi_*(E)}(E_*(X), E_*(Y)) & & \mathrm{Hom}_{E_*(E)}(E_*(X), E_*(E) \otimes_{\pi_*(E)} E_*(Y)) \arrow[ll, "\mathrm{adj}"'] \arrow[u, "(\Phi_{E,Y})_*"']
\end{tikzcd}$$

We know the left vertical map is an isomorphism by Theorem 5.13, and the bottom horizontal isomorphism is the forgetful-cofree adjunction (Proposition E.8) for A-graded left comodules over the dual E-Steenrod algebra. The right vertical arrow is a well-defined isomorphism, as $\Phi_{E,Y}$ is a homomorphism of A-graded left $E_*(E)$-comodules (Lemma 6.9), and in fact it is an isomorphism by Proposition 5.6, since $E_*(E)$ is flat and Y is cellular. Thus in order to see the top arrow is an isomorphism, it suffices to show that the diagram commutes. The left triangle clearly commutes; to see the right triangle commutes, recall that by how the how forgetful-cofree adjunction for left comodules over a Hopf algebroid is defined, that the bottom vertical arrow sends an A-graded homomorphism of left $E_*(E)$-comodules $\psi : E_*(X) \to E_*(E) \otimes_{\pi_*(E)} E_*(Y)$ to the composition

$$E_*(X) \xrightarrow{\psi} E_*(E) \otimes_{\pi_*(E)} E_*(Y) \xrightarrow{\pi_*(\mu)\otimes E_*(Y)} \pi_*(E) \otimes_{\pi_*(E)} E_*(Y) \xrightarrow{\cong} E_*(Y).$$

Thus, in order to show that this composition equals $\pi_*(\mu \otimes Y) \circ \Phi_{E,Y} \circ \psi$, it suffices to show the following diagram commutes:

$$\begin{tikzcd}
E_*(E) \otimes_{\pi_*(E)} E_*(Y) \arrow[r, "\pi_*(\mu)\otimes E_*(Y)"] \arrow[d, "\Phi_{E,Y}"'] & \pi_*(E) \otimes_{\pi_*(E)} E_*(Y) \arrow[d, "\cong"] \\
E_*(E \otimes Y) \arrow[r, "\pi_*(\mu\otimes Y)"] & E_*(Y)
\end{tikzcd}$$

Since all the arrows here are homomorphisms of abelian groups, in order to show the diagram commutes, it suffices to chase pure homogeneous tensors around. To that end, let $x : S^a \to E \otimes E$ and $y : S^b \to E \otimes Y$, and consider the following diagram exhibiting the two ways to chase $x \otimes y$ around:

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] & S^a \otimes S^b \arrow[r, "x\otimes y"] & E \otimes E \otimes E \otimes Y \arrow[r, "\mu\otimes E\otimes Y"] \arrow[d, "E\otimes\mu\otimes Y"'] & E \otimes E \otimes Y \arrow[d, "\mu\otimes Y"] \\
 & & E \otimes E \otimes Y \arrow[r, "\mu\otimes Y"] & E \otimes Y
\end{tikzcd}$$

The diagram commutes by associtiavity of μ. Thus, we have indeed shown that

$$E_*(-) : [X, E \otimes Y] \to \mathrm{Hom}_{E_*(E)}(E_*(X), E_*(E \otimes Y))$$

is an isomorphism of abelian groups, as desired. □

7. The Adams spectral sequence

Finally, we may construct the spectral sequence. Henceforth, we will assume the reader is familiar with the theory of spectral sequenes arising from unrolled exact couples, along with the notion of (conditional, strong) convergence of such spectral sequences to their (co)limits. The primary reference for these facts will be Boardman's paper [5] on conditionally convergent spectral sequences. When using any results from this reference, we will be sure to provide a proper citation. Note that Boardman works with $\mathbb{Z}$-graded groups, although everything he does carries through entirely the same with A-graded groups.

From now on, let (E, μ, e) be a monoid object and X and Y be objects in $\mathcal{SH}$.

7.1. Construction of the spectral sequence.

Definition 7.1. Let $\overline{E}$ be the fiber of the unit map $e : S \to E$ (Proposition A.4). Let $Y_0 := Y$ and $W_0 := E \otimes Y$. For $s > 0$, define

$$Y_s := \overline{E}^s \otimes Y, \qquad W_s := E \otimes Y_s = E \otimes \overline{E}^s \otimes Y,$$

where $\overline{E}^s$ denotes the s-fold tensor product $\overline{E} \otimes \cdots \otimes \overline{E}$. Then we get fiber sequences

$$Y_{s+1} \xrightarrow{i_s} Y_s \xrightarrow{j_s} W_s \xrightarrow{k_s} \Sigma Y_{s+1}$$

obtained by applying $- \otimes Y_s$ to the fiber sequence

$$\overline{E} \to S \xrightarrow{e} E \to \Sigma \overline{E}.$$

We can splice these sequences together to get the following diagram, which is called *the canonical Adams-resolution of Y*:

$$\begin{tikzcd}
\cdots \arrow[r] & Y_3 \arrow[r, "i_2"] \arrow[d, "j_3"] & Y_2 \arrow[r, "i_1"] \arrow[d, "j_2"] & Y_1 \arrow[r, "i_0"] \arrow[d, "j_1"] & Y_0 = Y \arrow[d, "j_0"] \\
& W_3 & W_2 \arrow[ul, dashed, "k_2"] & W_1 \arrow[ul, dashed, "k_1"] & W_0 \arrow[ul, dashed, "k_0"]
\end{tikzcd}$$

Here we are using dashed arrows to denote the (degree $-\mathbf{1}$) maps $k_s : W_s \to \Sigma Y_{s+1}$. In particular, the above diagram does not commute in any sense.

Now, by applying $[X, -]_*$ to the canonical E-Adams resolution of Y, we get an associated unrolled exact couple, and thus a spectral sequence:

Definition 7.2. Consider the canonical E-Adams resolution of Y from Lemma 7.3:

$$\begin{tikzcd}
\cdots \arrow[r] & Y_3 \arrow[r, "i_2"] \arrow[d, "j_3"] & Y_2 \arrow[r, "i_1"] \arrow[d, "j_2"] & Y_1 \arrow[r, "i_0"] \arrow[d, "j_1"] & Y_0 = Y \arrow[d, "j_0"] \\
& W_3 & W_2 \arrow[ul, dashed, "k_2"] & W_1 \arrow[ul, dashed, "k_1"] & W_0 \arrow[ul, dashed, "k_0"]
\end{tikzcd}$$

We can extend this diagram to the right by setting $Y_s = Y$, $W_s = 0$, and $i_s = \mathrm{id}_Y$ for $s < 0$. Then we may apply the functor $[X, -]_*$, and by Proposition 2.10, we obtain the following A-graded unrolled exact couple:

$$\begin{tikzcd}
\cdots \arrow[r] & {[X, Y_{s+2}]_*} \arrow[r, "i_{s+1}"] \arrow[d, "j_{s+2}"] & {[X, Y_{s+1}]_*} \arrow[r, "i_s"] \arrow[d, "j_{s+1}"] & {[X, Y_s]_*} \arrow[r, "i_{s-1}"] \arrow[d, "j_s"] & {[X, Y_{s-1}]_*} \arrow[r] \arrow[d, "j_{s-1}"] & \cdots \\
& {[X, W_{s+2}]_*} & {[X, W_{s+1}]_*} \arrow[ul, "\partial_{s+1}"] & {[X, W_s]_*} \arrow[ul, "\partial_s"] & {[X, W_{s-1}]_*} \arrow[ul, "\partial_{s-1}"] &
\end{tikzcd}$$

where here we are being abusive and writing $i_s : [X, Y_{s+1}]_* \to [X, Y_s]_*$ and $j_s : [X, Y_s]_* \to [X, W_s]_*$ to denote the pushforward maps induced by $i_s : Y_{s+1} \to Y_s$ and $j_s : Y_s \to W_s$, respectively. Each i_s, j_s, and ∂_s are A-graded homomorphisms of degrees 0, 0, and $-\mathbf{1}$, respectively.

In [5, §0], it is described how we may associate a $\mathbb{Z} \times A$-graded spectral sequence $r \mapsto (E_r^{*,*}(X,Y), d_r)$ to the above A-graded unrolled exact couple, where d_r has $\mathbb{Z} \times A$-degree $(r, -\mathbf{1})$. We call this spectral sequence the *E-Adams spectral sequence for the computation of* $[X,Y]_*$.

For those who would rather not lose themselves in Boardman's document, we give a brief unravelling of how it applies to the present situation. Given some $s \in \mathbb{Z}$ and some $r \geq 1$, we may define the following A-graded subgroups of $[X, W_s]_*$:

$$Z_r^s := \partial_s^{-1}(\operatorname{im}[i^{(r-1)} : [X, Y_{s+r}]_* \to [X, Y_{s+1}]_*])$$

and

$$B_r^s := j_s(\ker[i^{(r-1)} : [X, Y_s]_* \to [X, Y_{s-r+1}]_*]),$$

where we adopt the convention that $i^{(0)}$ is simply the identity. This yields an infinite sequence of inclusions

$$0 = B_1^s \subseteq B_2^s \subseteq B_3^s \subseteq \cdots \subseteq \operatorname{im} j_s = \ker \partial_s \subseteq \cdots \subseteq Z_3^s \subseteq Z_2^s \subseteq Z_1^s = [X, W_s]_*.$$

Then for $r \geq 1$, we define E_r^s to be the A-graded quotient group

$$E_r^s := Z_r^s / B_r^s.$$

Thus taking the direct sum of all the E_r^s's yields the r^{th} page of the spectral sequence

$$E_r := \bigoplus_{s \in \mathbb{Z}} E_r^s,$$

which is a $\mathbb{Z} \times A$-graded abelian group.

The differential $d_r : E_r \to E_r$ is a map of $\mathbb{Z} \times A$-degree $(r, \mathbf{1})$, and is constructed as follows: an element of $E_r^s = Z_r^s / B_r^s$ is a coset represented by some $x \in Z_r^s$, so that $\partial_s(x) = i^{(r-1)}(y)$ for some $y \in [X, Y_{s+r}]_*$. Then we define $d_r([x])$ to be the coset $[j_{s+r}(y)]$ in Z_r^{s+r}/B_r^{s+r}.

In the case $r = 1$, since $B_1^s = 0$ and $Z_1^s = [X, W_s]_*$, we have that $E_1^s = [X, W_s]_*$, and given some $x \in E_1^s = [X, W_s]_*$, the differential d_1 is given by $d_1(x) = j_{s+1}(\partial_s(x))$, so that $d_1 = j \circ \partial$. Furthermore, since the unrolled exact couple which yields the spectral sequence vanishes on its negative terms, we have that $E_r^{s,a}(X,Y) = 0$ for $s < 0$. In particular, the E-Adams spectral sequence is a *half-plane spectral sequence with entering differentials*, in the sense of [5, §7].

Showing in explicit detail that all of these definitions make sense and are well-defined is relatively straightforward. Furthermore, one may check that that $d_r \circ d_r = 0$, and that

$$\ker d_r^s / \operatorname{im} d_r^s = \frac{Z_{r+1}^s / B_r^s}{B_{r+1}^s / B_r^s} \cong Z_{r+1}^s / B_{r+1}^s = E_{r+1}^s.$$

Above we constructed the spectral sequence by means of the "canonical" E-Adams resolution of Y, but one may more generally pursue the notion of *E-Adams resolutions* of the object Y, for which the canonical Adams resolution constructed above will be an example. We do not explore this generality here (although one certainly could); these are useful when one wants to construct an Adams resolution from an algebraic resolution of $E_*(Y)$, or by modifying an Adams resolution for some other object. One may find different notions of what exactly constitutes an Adams resolution in the literature (for example, see [26, Definition 2.2.1] or [28, Definition 11.3.1]), and they will always be defined so that the E-Adams spectral sequence for $[X,Y]_*$ is independent of the choice of Adams resolution for Y, at least from its E_2 page onwards. One important condition (or definitional consequence) one will always find for an E-Adams resolution is that the i's must vanish in E-homology. We can show that the canonical E-Adams resolution we have constructed satisfies this property:

Lemma 7.3. *Let i_s and j_s be as in Definition 7.1. Then the maps $j_s : Y_s \to W_s$ induce split monomorphisms $E_*(j_s)$ on E-homology, so that in particular the maps $i_s : Y_{s+1} \to Y_s$ vanish in E-homology, i.e., $E_*(i_s)$ is the zero map.*

Proof. First, note that since

$$Y_{s+1} \xrightarrow{i_s} Y_s \xrightarrow{j_s} W_s \xrightarrow{k_s} \Sigma Y_{s+1}$$

is a distinguished triangle and $\mathcal{SH}$ is tensor triangulated, there is a distinguished triangle of the form

$$E \otimes Y_{s+1} \xrightarrow{E\otimes i_s} E \otimes Y_s \xrightarrow{E\otimes j_s} E \otimes W_s \to \Sigma(E \otimes Y_{s+1}).$$

Thus, applying $\pi_*(-) \cong [S, -]_*$ to the triangle yields that the following sequence is exact (see Proposition A.2 for details):

$$E_*(Y_{s+1}) \xrightarrow{E_*(i_s)} E_*(Y_s) \xrightarrow{E_*(j_s)} E_*(W_s).$$

Now, it is straightforward to verify by construction that j_s is the map $e \otimes Y_s : Y_s \to E \otimes Y_s = W_s$. Thus, by unitality of μ, we have that $E \otimes j_s : E \otimes Y_s \to E \otimes W_s$ is a split monomorphism, with right inverse $\mu \otimes Y_s : E \otimes W_s = E \otimes E \otimes Y_s \to E \otimes Y_s$. Then since any functor preserves split monomorphisms, it follows that $E_*(j_s) = \pi_*(E \otimes j_s)$ is likewise a split monomorphism, so that in particular $E_*(j_s)$ is injective. Thus $\operatorname{im} E_*(i_s) = \ker E_*(j_s) = 0$, so that i_s is indeed the zero map, as desired. □

7.2. **The E_2 page.** Now, we would like to characterize the E_2 page of the spectral sequence in terms of something more concrete. Namely, we will characterize the E_2 page in terms of Ext of comodules over the dual E-Steenrod algebra. For a quick review of Ext in an abelian category and derived functors, see Appendix D. The goal of this subsection will be to prove the following theorem:

Theorem 7.4. *Let (E, μ, e) be a commutative monoid object, and X and Y objects in $\mathcal{SH}$. Suppose further that:*

- *E is flat (Definition 6.5) and cellular (Definition 3.1),*
- *X is cellular and $E_*(X)$ is a graded projective left $\pi_*(E)$-module (via Proposition 4.3),*
- *Y is cellular.*

Then the non-vanishing entries of the second page of the E-Adams spectral sequence for the computation of $[X, Y]_*$ (Definition 7.2) are the Ext *groups of A-graded left comodules over the anticommutative Hopf algebroid structure on the dual E-Steenrod algebra (Proposition 6.7), i.e., we have the following isomorphisms for all $s \geq 0$ and $a \in A$:*

$$E_2^{s,a}(X, Y) \cong \operatorname{Ext}^{s,a+\mathbf{s}}_{E_*(E)}(E_*(X), E_*(Y)) := \operatorname{Ext}^{s}_{E_*(E)}(E_*(X), E_{*+a+\mathbf{s}}(Y)).$$

Proof. By Proposition 7.8 below, for each $s \geq 0$ and $a \in A$, $E_2^{s,a}(X, Y)$ is isomorphic to the s^{th} cohomology group of the cochain complex obtained by applying $F := \operatorname{Hom}^{a+\mathbf{s}}_{E_*(E)}(E_*(X), -)$ to the complex

$$0 \longrightarrow E_*(W_0) \xrightarrow{E_*(\delta_0)} E_*(\Sigma W_1) \xrightarrow{E_*(\delta_1)} E_*(\Sigma^2 W_2) \xrightarrow{E_*(\delta_2)} E_*(\Sigma^3 W_3) \longrightarrow \cdots.$$

Furthermore, by Lemma 7.7, this complex is an F-acyclic resolution of $E_*(Y)$ (Definition D.4). Thus, since the category of $E_*(E)$-comodules is an abelian category with enough injectives (Proposition E.9), we have by Proposition D.5 that

$$E_2^{s,a}(X, Y) \cong R^s\operatorname{Hom}^{a+\mathbf{s}}_{E_*(E)}(E_*(X), -)(E_*(Y)) = \operatorname{Ext}^{s,a+\mathbf{s}}(E_*(X), E_*(Y)),$$

as desired. □

As a result of this theorem, the spectral sequence is often shifted, by re-defining

$$E_r^{s,a}(X, Y)^{\text{new}} := E_r^{s,a+\mathbf{s}}(X, Y),$$

in which case the given isomorphism characterizing the E_2 page is strictly degree-preserving. This is in fact the standard convention for the classical stable homotopy category. We leave it to the reader to unravel what the differential d_2 corresponds to under this identification. The remainder of this subsection is devoted to proving Lemma 7.7 and Proposition 7.8. To start, we establish the following convention:

Definition 7.5. Given some (nonnegative integer) $n \geq 0$, define natural isomorphisms $\nu_X^n : \Sigma^{\mathbf{n}} X \to \Sigma^n X$ inductively, by setting $\nu_X^0 := \lambda_X$, $\nu_X^1 := \nu_X^{-1}$, and supposing ν_X^{n-1} has been defined for some $n > 1$, define ν_X^n to be the composition

$$\nu_X^n : \Sigma^{\mathbf{n}} X = S^{\mathbf{n}} \otimes X \xrightarrow{\phi_{\mathbf{n-1},\mathbf{1}} \otimes X} S^{\mathbf{n-1}} \otimes S^{\mathbf{1}} \otimes X \xrightarrow{S^{\mathbf{n-1}} \otimes \nu_X^{-1}} S^{\mathbf{n-1}} \Sigma X \xrightarrow{\nu_{\Sigma X}^{\mathbf{n-1}}} \Sigma^n X.$$

By induction, naturality of ν, and functoriality of $- \otimes -$, these isomorphisms are clearly natural in X.

Lemma 7.6. *Suppose E and Y are cellular. Then for all $s \in \mathbb{Z}$, the objects Y_s and W_s from the canonical E-Adams resolution of Y (Definition 7.1) are cellular.*

Proof. Unravelling definitions, for $s < 0$, $W_s = 0$ and $Y_s = Y$, which are both cellular.[6] For $s \geq 0$, we have $W_s = E \otimes Y_s$, so that by cellularity of E and Lemma 3.4, it suffices to show that Y_s is cellular for $s \geq 0$. We know $Y_0 = Y$ is cellular by definition. For $s > 0$, Y_s is the tensor product $\overline{E}^s \otimes Y$, where $\overline{E}$ fits into the distinguished triangle

$$\overline{E} \to S \xrightarrow{e} E \to \Sigma \overline{E}.$$

By the definition of cellularity, $\overline{E}$ is cellular since S and E are. Thus $\overline{E}^s \otimes Y$ is cellular by Lemma 3.4, as it is a tensor product of cellular objects in $\mathcal{SH}$. $\square$

Lemma 7.7. *Let (E, μ, e) be a flat (Definition 6.5) and cellular (Definition 3.1) commutative monoid object and X and Y cellular objects in $\mathcal{SH}$, and for $s \geq 0$ define Y_s and W_s as in Definition 7.1. In particular, for each $s \geq 0$, $W_s = E \otimes Y_s$ and we have distinguished triangles*

$$Y_{s+1} \xrightarrow{i_s} Y_s \xrightarrow{j_s} W_s \xrightarrow{k_s} \Sigma Y_{s+1}.$$

Then if $E_*(X)$ is a graded projective (Definition B.15) left $\pi_*(E)$-module (via Proposition 4.3) then the sequence

$$0 \to E_*(Y) \xrightarrow{E_*(j_0)} E_*(W_0) \xrightarrow{E_*(\delta_0)} E_*(\Sigma W_1) \xrightarrow{E_*(\delta_1)} E_*(\Sigma^2 W_2) \xrightarrow{E_*(\delta_2)} E_*(\Sigma^3 W_3) \to \cdots$$

is an F-acyclic resolution (Definition D.4) of $E_*(Y)$ in $E_*(E)$-$\mathbf{CoMod}^A$ for

$$F = \mathrm{Hom}^a_{E_*(E)}(E_*(X), -)$$

for all $a \in A$, where δ_s is the composition

$$\Sigma^s W_s \xrightarrow{\Sigma^s k_s} \Sigma^{s+1} Y_{s+1} \xrightarrow{\Sigma^{s+1} j_{s+1}} \Sigma^{s+1} W_{s+1}.$$

Proof. By Lemma 7.6, each W_s is cellular, so that furthermore $\Sigma^s W_s \cong S^{\mathbf{s}} \otimes W_s$ is cellular for each $s \geq 0$, by Lemma 3.4. Thus, the sequence does indeed live in $E_*(E)$-$\mathbf{CoMod}^A$ by Proposition 6.8, as desired. Next, we claim that $E_*(\Sigma^s W_s)$ is an F-acyclic object for each $s \geq 0$, i.e., that

$$\mathrm{Ext}^{n,a}_{E_*(E)}(E_*(X), E_*(\Sigma^s W_s)) = \mathrm{Ext}^n_{E_*(E)}(E_*(X), E_{*+a}(\Sigma^s W_s)) = 0$$

[6] 0 is cellular because it is the cofiber of the identity on S by axiom TR1 for a triangulated category (Definition 2.1), i.e., there is a distinguished triangle $S \to S \to 0 \to \Sigma S$.

for all $n > 0$, $s \geq 0$, and $a \in A$. Note that we have an A-graded isomorphism of left $E_*(E)$-comodules:

$$\begin{tikzcd}
E_*(E) \otimes_{\pi_*(E)} E_{*+a}(\Sigma^s Y_s) \arrow[r, equal] & E_*(E) \otimes_{\pi_*(E)} E_{*+a}(\Sigma^s Y_s) \arrow[d, "\Phi_{E,\Sigma^s Y_s}"] & \\
& E_*(E \otimes \Sigma^s Y_s) \arrow[d, "E_*(E \otimes (\nu^s_{Y_s})^{-1})"] & \\
& E_*(E \otimes S^{\mathbf{s}} \otimes Y_s) \arrow[d, "E_*(\tau \otimes Y_s)"] & \\
& E_*(S^{\mathbf{s}} \otimes E \otimes Y_s) \arrow[d, "E_*(\nu^s_{E \otimes Y_s})"] & \\
& E_*(\Sigma^s(E \otimes Y_s)) \arrow[r, equal] & E_*(\Sigma^s W_s)
\end{tikzcd}$$

where $\Phi_{E,\Sigma^s Y}$ is an A-graded isomorphism of abelian groups by Proposition 5.6, and furthermore an isomorphism of $E_*(E)$-comodules by Lemma 6.9. Every other arrow is an isomorphism of $E_*(E)$-comodules by functoriality of $E_*(-) : \mathcal{SH}\text{-}\mathbf{Cell} \to E_*(E)\text{-}\mathbf{CoMod}^A$. Thus, since $E_*(\Sigma^s W_s)$ is isomorphic to $E_*(E) \otimes_{\pi_*(E)} E_{*+a}(\Sigma^s Y_s)$ in $E_*(E)\text{-}\mathbf{CoMod}^A$, and in particular since $\mathrm{Ext}^n_{E_*(E)}(E_*(X), -)$ is a functor, we have

$$\mathrm{Ext}^n_{E_*(E)}(E_*(X), E_{*+a}(\Sigma^s W_s)) \cong \mathrm{Ext}^n_{E_*(E)}(E_*(X), E_*(E) \otimes_{\pi_*(E)} E_{*+a}(\Sigma^s Y_s)).$$

Yet, $E_*(E) \otimes_{\pi_*(E)} E_{*+a}(\Sigma^s Y_s)$ is a co-free $E_*(E)$-comodule (Proposition E.8), in which case since $E_*(X)$ is graded projective as an object in $\pi_*(E)\text{-}\mathbf{Mod}^A$, we have that

$$\mathrm{Ext}^{n,a}_{E_*(E)}(E_*(X), E_*(E) \otimes_{\pi_*(E)} E_{*+a}(\Sigma^s Y_s)) = 0,$$

by Proposition E.10.

Finally, it remains to show that the sequence is exact. To that end, first note that by induction on axiom TR4 for a triangulated category and the fact that distinguished triangles are exact (Proposition A.2), the following sequence in $\mathcal{SH}$ is exact (since a sequence clearly remains exact even after changing the signs of its maps):

$$\Sigma^s Y_s \xrightarrow{\Sigma^s j_s} \Sigma^s W_s \xrightarrow{\Sigma^s k_s} \Sigma^{s+1} Y_{s+1} \xrightarrow{\Sigma^{s+1} i_s} \Sigma^{s+1} Y_s \xrightarrow{\Sigma^{s+1} j_s} \Sigma^{s+1} W_s$$

(see Definition A.1 for the definition of an exact triangle in an additive category). Furthermore, since $\mathcal{SH}$ is tensor triangulated, the sequence remains exact after applying $E \otimes -$ (see Proposition A.11 for details), so that taking E-homology yields the following exact sequence of homology groups:

$$E_*(\Sigma^s Y_{s+1}) \xrightarrow{E_*(\Sigma^s i_s)} E_*(\Sigma^s Y_s) \xrightarrow{E_*(\Sigma^s j_s)} E_*(\Sigma^s W_s) \xrightarrow{E_*(\Sigma^s k_s)} E_*(\Sigma^{s+1} Y_{s+1}) \xrightarrow{E_*(\Sigma^{s+1} i_s)} E_*(\Sigma^{s+1} Y_s).$$

Then since $E_*(i_s) : E_*(Y_{s+1}) \to E_*(Y_s)$ is the zero map (by Lemma 7.3) and we have natural isomorphisms

$$E_*(\Sigma^t X) \xrightarrow{\nu^t_X} E_*(\Sigma^{\mathbf{t}} X) \xrightarrow{t^{\mathbf{t}}_X} E_{*-\mathbf{t}}(X)$$

(the first from Definition 7.5 and the latter from Lemma 4.4), we have that $E_*(\Sigma^t i_s) : E_*(\Sigma^t Y_{s+1}) \to E_*(\Sigma^t Y_s)$ is the zero map for all $t \in \mathbb{Z}$, so that in particular the above exact sequence splits to yield the short exact sequence

$$0 \to E_*(\Sigma^s Y_s) \xrightarrow{E_*(\Sigma^s j_s)} E_*(\Sigma^s W_s) \xrightarrow{E_*(\Sigma^s k_s)} E_*(\Sigma^{s+1} Y_{s+1}) \to 0.$$

Then we may splice these sequences together for $s \geq 0$ to yield the following diagram:

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & E_*(Y) \arrow[r, "E_*(j_0)"] & E_*(W_0) \arrow[rr, "E_*(\delta_0)"] \arrow[dr, "E_*(k_0)"'] & & E_*(\Sigma W_1) \arrow[rr, "E_*(\delta_1)"] \arrow[dr, "E_*(\Sigma k_1)"'] & & E_*(\Sigma^2 W_2) \arrow[r] & \cdots \\
& & & E_*(\Sigma Y_1) \arrow[ur, "E_*(\Sigma j_1)"'] & & E_*(\Sigma^2 Y_2) \arrow[ur, "E_*(\Sigma^2 j_2)"'] & &
\end{tikzcd}$$

It is straightforward to check the top row is exact by exactness of the short exact sequences, as desired. ☐

Proposition 7.8. *Let (E, μ, e) be a commutative monoid object, and X and Y objects in $\mathcal{SH}$. Suppose further that:*

- *E is flat (Definition 6.5) and cellular (Definition 3.1),*
- *X is cellular, and $E_*(X)$ is a graded projective left $\pi_*(E)$-module (via Proposition 4.3), and*
- *Y is cellular.*

Then for all $s \in \mathbb{Z}$ and $a \in A$, the line in the first page of the E-Adams spectral sequence for the computation of $[X, Y]_*$ (Definition 7.2)

$$0 \to E_1^{0,a+\mathbf{s}}(X, Y) \xrightarrow{d_1} E_1^{1,a+\mathbf{s}-\mathbf{1}}(X, Y) \xrightarrow{d_1} E_1^{2,a+\mathbf{s}-\mathbf{2}}(X, Y) \to \cdots \to E_1^{s,a}(X, Y) \to \cdots$$

is isomorphic to the complex obtained by applying $\mathrm{Hom}^{a+\mathbf{s}}_{E_*(E)}(E_*(X), -)$ to the complex of A-graded left $E_*(E)$-comodules

$$0 \to E_*(W_0) \xrightarrow{E_*(\delta_0)} E_*(\Sigma W_1) \xrightarrow{E_*(\delta_1)} E_*(\Sigma^2 W_2) \to \cdots \to E_*(\Sigma^s W_s) \to \cdots$$

from Lemma 7.7.

Proof. By Lemma 7.6, since E and Y are cellular, W_t is as well for each $t \geq 0$. Furthermore, for $t > 0$, we have isomorphisms

$$S^{\mathbf{t}} \otimes W_t \xrightarrow{\nu^t_{W_t}} \Sigma^t W_t,$$

and by Lemma 3.4, the object $S^{\mathbf{t}} \otimes W_t$ is cellular since $S^{\mathbf{t}}$ and W_t are. Hence, by Proposition 6.8, the complex

$$0 \to E_*(W_0) \xrightarrow{E_*(\delta_0)} E_*(\Sigma W_1) \xrightarrow{E_*(\delta_1)} E_*(\Sigma^2 W_2) \to \cdots \to E_*(\Sigma^s W_s) \to \cdots$$

actually lives in $E_*(E)$-$\mathbf{CoMod}^A$, as desired. Now, let $t \geq 0$, and consider the following diagram:

$$\begin{tikzcd}
{[X, W_t]_{a+\mathbf{s}-\mathbf{t}}} \arrow[d, "(k_t)_*"'] & {[X, \Sigma^{\mathbf{t}} W_t]_{a+\mathbf{s}}} \arrow[l, "s^{\mathbf{t}}_{X,W_t}"'] \arrow[r, "(\nu^t_{W_t})_*"] \arrow[d, dashed, "(\Sigma^{\mathbf{t}} k_t)_*"] & {[X, \Sigma^t W_t]_{a+\mathbf{s}}} \arrow[dd, "(\Sigma^t k_t)_*"] \arrow[dddd, bend left, "(\delta_t)_*"] \\
{[X, \Sigma Y_{t+1}]_{a+\mathbf{s}-\mathbf{t}}} \arrow[d, "(\nu_{Y_{t+1}})_*"'] & {[X, \Sigma^{\mathbf{t}} \Sigma Y_{t+1}]_{a+\mathbf{s}}} \arrow[l, "s^{\mathbf{t}}_{X,\Sigma Y_{t+1}}"'] \arrow[d, dashed, "(\Sigma^{\mathbf{t}} \nu_{Y_{t+1}})_*"] \arrow[dr, "(\nu^t_{\Sigma Y_{t+1}})_*"] & \\
{[X, \Sigma^{\mathbf{1}} Y_{t+1}]_{a+\mathbf{s}-\mathbf{t}}} \arrow[d, "s^{\mathbf{1}}_{X,Y_{t+1}}"'] & {[X, \Sigma^{\mathbf{t}} \Sigma^{\mathbf{1}} Y_{t+1}]_{a+\mathbf{s}}} \arrow[l, "s^{\mathbf{t}}_{X,\Sigma^{\mathbf{1}} Y_{t+1}}"'] & {[X, \Sigma^{t+1} Y_{t+1}]_{a+\mathbf{s}}} \arrow[dd, "(\Sigma^{t+1} j_{t+1})_*"] \\
{[X, Y_{t+1}]_{a+\mathbf{s}-\mathbf{t}-\mathbf{1}}} \arrow[d, "(j_{t+1})_*"'] & {[X, \Sigma^{\mathbf{t}+\mathbf{1}} Y_{t+1}]_{a+\mathbf{s}}} \arrow[l, "s^{\mathbf{t}+\mathbf{1}}_{X,Y_{t+1}}"'] \arrow[u, dashed, "(\phi_{\mathbf{t},\mathbf{1}} \otimes Y_{t+1})_*"] \arrow[ur, "(\nu^{t+1}_{Y_{t+1}})_*"'] \arrow[d, dashed, "(\Sigma^{\mathbf{t}+\mathbf{1}} j_{t+1})_*"] & \\
{[X, W_{t+1}]_{a+\mathbf{s}-\mathbf{t}-\mathbf{1}}} & {[X, \Sigma^{\mathbf{t}+\mathbf{1}} W_{t+1}]_{a+\mathbf{s}}} \arrow[l, "s^{\mathbf{t}+\mathbf{1}}_{X,W_{t+1}}"'] \arrow[r, "(\nu^{t+1}_{W_{t+1}})_*"] & {[X, \Sigma^{t+1} W_{t+1}]_{a+\mathbf{s}}}
\end{tikzcd}$$

where here the $s^a_{X,Y} : [X, \Sigma^a Y]_* \cong [X, Y]_{*-a}$'s are the natural isomorphisms from Definition 2.9. By unravelling definitions, we have the top left object is $E_1^{t,a+\mathbf{s}-\mathbf{t}}(X, Y)$ and the bottom left object is $E_1^{t+1,a+\mathbf{s}-\mathbf{t}-\mathbf{1}}$, and the vertical left composition in the above diagram is the differential d_1 between them. The first, second, and fourth rectangles from the top on the left rectangle commute by naturality of the s^a's. Furthermore, a simple diagram chase and coherence of the ϕ's (Remark 2.4) yields that the third rectangle on the left commutes. The trapezoids on the right commute by naturality of ν^t and ν^{t+1}. Finally, the middle right triangle commutes by how we defined ν^{t+1} in terms of ν^t.

Now, consider the following diagram:

$$\begin{tikzcd}
E_1^{t,a+\mathbf{s}-\mathbf{t}}(X,Y) \arrow[r, "d_1"] \arrow[d, "(s^{\mathbf{t}}_{X,W_t})^{-1}"'] & E_1^{t+1,a+\mathbf{s}-\mathbf{t}-\mathbf{1}}(X,Y) \arrow[d, "(s^{\mathbf{t+1}}_{X,W_{t+1}})^{-1}"] \\
{[X, \Sigma^{\mathbf{t}} W_t]_{a+\mathbf{s}}} \arrow[d, "(\nu^t_{W_t})_*"'] & {[X, \Sigma^{\mathbf{t+1}} W_{t+1}]_{a+\mathbf{s}}} \arrow[d, "(\nu^{t+1}_{W_{t+1}})_*"] \\
{[X, \Sigma^t W_t]_{a+\mathbf{s}}} \arrow[r, "(\delta_t)_*"] \arrow[d, "E_*(-)"'] & {[X, \Sigma^{t+1} W_{t+1}]_{a+\mathbf{s}}} \arrow[d, "E_*(-)"] \\
\mathrm{Hom}_{E_*(E)}(E_*(\Sigma^{a+\mathbf{s}} X), E_*(\Sigma^t W_t)) \arrow[r, "E_*(\delta_t)"] \arrow[d, "((t_X^{a+\mathbf{s}})^{-1})^*"'] & \mathrm{Hom}_{E_*(E)}(E_*(\Sigma^{a+\mathbf{s}} X), E_*(\Sigma^{t+1} W_{t+1})) \arrow[d, "((t_X^{a+\mathbf{s}})^{-1})^*"] \\
\mathrm{Hom}^{a+\mathbf{s}}_{E_*(E)}(E_*(X), E_*(\Sigma^t W_t)) \arrow[r, "E_*(\delta_t)"] & \mathrm{Hom}^{a+\mathbf{s}}_{E_*(E)}(E_*(X), E_*(\Sigma^{t+1} W_{t+1}))
\end{tikzcd}$$

where here the maps $t_X^{a+\mathbf{s}} : E_*(\Sigma^a) \to E_{*-a}(X)$ are the $E_*(E)$-comodule isomorphisms from Lemma 6.10. We have just shown the top region commutes. Furthermore, since X and $\Sigma^t W_t$ are cellular for all $t \geq 0$, the arrows labelled $E_*(-)$ are well-defined, and they clearly make the middle rectangle commute (a simple diagram chase suffices). The bottom rectangle also clearly commutes, Thus, it suffices to show that the maps labelled $E_*(-)$ are isomorphisms. To that end, consider the following diagram:

$$\begin{tikzcd}
{[X, \Sigma^t W_t]_{a+\mathbf{s}}} \arrow[r, "E_*(-)"] \arrow[d, "f_*"'] & \mathrm{Hom}_{E_*(E)}(E_*(\Sigma^{a+\mathbf{s}} X), E_*(\Sigma^t W_t)) \arrow[d, "E_*(f)_*"] \\
{[X, E \otimes \Sigma^{\mathbf{t}} Y_t]_{a+\mathbf{s}}} \arrow[r, "E_*(-)"] & \mathrm{Hom}_{E_*(E)}(E_*(\Sigma^{a+\mathbf{s}} X), E_*(E \otimes \Sigma^{\mathbf{t}} Y_t))
\end{tikzcd}$$

where here $f : \Sigma^t W_t \to E \otimes \Sigma^{\mathbf{t}} Y_t$ is the isomorphism

$$\Sigma^t W_t \xrightarrow{\nu^t_W} \Sigma^{\mathbf{t}} W_t = S^{\mathbf{t}} \otimes E \otimes Y_t \xrightarrow{\tau \otimes Y_t} E \otimes S^{\mathbf{t}} \otimes Y_t = E \otimes \Sigma^{\mathbf{t}} Y_t.$$

The bottom horizontal arrow is an isomorphism by Theorem 6.11. Thus, the top horizontal arrow is an isomorphism, as desired. Showing

$$E_*(-) : [X, \Sigma^{t+1} W_{t+1}]_{a+\mathbf{s}} \to \mathrm{Hom}_{E_*(E)}(E_*(\Sigma^{a+\mathbf{s}} X), E_*(\Sigma^{t+1} W_{t+1}))$$

is an isomorphism is entirely analagous. Thus, for each $t \geq 0$, we have constructed isomorphisms

$$E^{t,a+\mathbf{s}-\mathbf{t}}(X, Y) \xrightarrow{\cong} \mathrm{Hom}^{a+\mathbf{s}}_{E_*(E)}(E_*(X), E_*(\Sigma^t W_t))$$

such that the following diagram commutes:

$$\begin{tikzcd}
E^{t,a+\mathbf{s}-\mathbf{t}}(X,Y) \arrow[r, "d_1"] \arrow[d, "\cong"'] & E^{t+1,a+\mathbf{s}-\mathbf{t}-\mathbf{1}}(X,Y) \arrow[d, "\cong"] \\
\mathrm{Hom}^{a+\mathbf{s}}_{E_*(E)}(E_*(X), E_*(\Sigma^t W_t)) \arrow[r, "\mathrm{Hom}^{a+\mathbf{s}}_{E_*(E)}(E_*(X), E_*(\delta_t))"] & \mathrm{Hom}^{a+\mathbf{s}}_{E_*(E)}(E_*(X), E_*(\Sigma^{t+1} W_{t+1}))
\end{tikzcd}$$

Hence, we have proven the desired result. □

7.3. **Convergence of the spectral sequence.** In this subsection, we briefly sketch some converge properties of the spectral sequence. Boardman already works quite generally in [5], so most of this is simply a review of the material contained within. From now on, we assume familiarity with derived limits of (A-graded) abelian groups (see Boardman §1), filtered (A-graded) groups (see Boardman §2), convergence of spectral sequences (Boardman Definition 5.2) and conditional convergence of a spectral sequence associated to an unrolled exact couple (Boardman Definition 5.10). We adopt his notation, writing

$$E^s_\infty(X,Y) := \left(\bigcap_{r=1}^{\infty} Z^s_r\right) \Big/ \left(\bigcup_{r=1}^{\infty} B^s_r\right) \qquad \text{and} \qquad RE_\infty(X,Y) := \operatorname*{Rlim}_r Z^s_r$$

to denote the E_∞-term and the derived E_∞-term of the spectral sequence, respectively.

Ideally, the E-Adams spectral sequence for $[X,Y]_*$ would give us information which allows us to compute the group $[X,Y]_*$. Note that $[X,Y]_*$ is the colimit of the unrolled exact couple which determines the spectral sequence, as $Y_s = Y$ for $s < 0$. Furthermore, since $(E_r(X,Y), d_r)$ is a half-plane spectral sequence with entering differentials, we may apply the results from [5, §7], where suitable conditions under which the spectral sequence converges to the colimit $[X,Y]_*$ are described (in particular, see Theorem 7.3 there). Unfortunately, in practice, the conditions outlined there are not usually satisfied for this spectral sequence, namely, in order for the spectral sequence to converge to $[X,Y]_*$, we must have that $\lim_s [X,Y_s]_* = 0$. There is no reason to believe this would be satisfied, so we must take an alternative approach. Following Section 5 of Bousfield's seminal paper [6], we can instead set up the spectral sequence by means of a tower *under* Y. First, we must define the *E-nilpotent completion of* Y:

Definition 7.9 ([6, pgs. 272–273])**.** Let (E,μ,e) be a monoid object in $\mathcal{SH}$, and Y any object. Write $\overline{E}$ for the fiber (Proposition A.4) of the unit $S \xrightarrow{e} E$, so we have a distinguished triangle

$$\overline{E} \to S \xrightarrow{e} E \to \Sigma\overline{E}.$$

Set $Y_0 := Y$ and $W_0 := Y \otimes E$, and for $s > 0$ define $Y_s := Y \otimes \overline{E}^s$ and $W_s := Y_s \otimes E$. Then since $\mathcal{SH}$ is tensor triangulated, for each $s \geq 0$ we may tensor the above sequence with Y_s on the right, which yields the following distinguished triangle

$$Y_{s+1} \xrightarrow{i} Y_s \xrightarrow{j} W_s \xrightarrow{k} \Sigma Y_{s+1}.$$

Then for $s \geq 0$, define Y/Y^s (up to non-canonical isomorphism) to be the cofiber of $i^s : Y_s \to Y_0 = Y$ (so in particular we may take $Y/Y_1 = E \otimes Y$ and $Y/Y_0 = 0$), so we have a distinguished triangle

$$Y_s \xrightarrow{i^s} Y \xrightarrow{b} Y/Y_s \xrightarrow{c} \Sigma Y_s.$$

Then for each $s \geq 0$, by the octahedral axiom (axiom TR5) for a triangulated category applied to the triangles

$$Y_{s+1} \xrightarrow{i} Y_s \xrightarrow{j} W_s \xrightarrow{k} \Sigma Y_{s+1}$$
$$Y_s \xrightarrow{i^s} Y \xrightarrow{b} Y/Y_s \xrightarrow{c} \Sigma Y_s$$
$$Y_{s+1} \xrightarrow{i^{s+1}} Y \xrightarrow{b} Y/Y_{s+1} \xrightarrow{c} \Sigma Y_{s+1},$$

there exists a distinguished triangle

$$W_s \xrightarrow{p} Y/Y_{s+1} \xrightarrow{q} Y/Y_s \xrightarrow{r} \Sigma W_s \tag{8}$$

which makes the following diagram commute:

$$\begin{tikzcd}
Y_{s+1} \arrow[rr, "i^{s+1}"] \arrow[dr, "i"'] & & Y \arrow[rr, "b"] \arrow[dr, "b"] & & Y/Y_s \arrow[rr, "r"] \arrow[dr, "c"] & & \Sigma W_s \\
& Y_s \arrow[ur, "i^s"'] \arrow[dr, "j"'] & & Y/Y_{s+1} \arrow[ur, "q"'] \arrow[dr, "c"] & & \Sigma Y_s \arrow[ur, "\Sigma j"'] & \\
& & W_s \arrow[ur, "p"] \arrow[rr, "k"] & & \Sigma Y_{s+1} \arrow[ur, "\Sigma i"'] & &
\end{tikzcd} \tag{9}$$

The triangles from (8) for $s \geq 0$ may be spliced together to yield a tower $\{Y/Y_s\}_s$ under Y:

$$\begin{tikzcd}
Y \arrow[r] & \cdots \arrow[r] & Y/Y_3 \arrow[r, "q"] \arrow[d, dashed, "r"] & Y/Y_2 \arrow[r, "q"] \arrow[d, dashed, "r"] \arrow[dl, leftarrow, "p"'] & Y/Y_1 \arrow[r, "q"] \arrow[d, dashed, "r"] \arrow[dl, leftarrow, "p"'] & Y/Y_0 \arrow[r, equal] \arrow[d, dashed, "r"] \arrow[dl, leftarrow, "p"'] & 0 \\
& & W_3 & W_2 & W_1 & W_0 &
\end{tikzcd}$$

where here the dashed arrows are really (degree $-\mathbf{1}$) maps $Y/Y_s \to \Sigma W_s$. The fact that this is a tower under Y follows from diagram (9), which tells us that $Y \xrightarrow{b} Y/Y_s$ factors as $Y \xrightarrow{b} Y/Y_{s+1} \xrightarrow{q} Y/Y_s$. We define the *$E$-nilpotent completion of Y* to be the object $Y^{\wedge}_E$ (defined up to non-canonical isomorphism) obtained as the homotopy limit of this tower (Definition A.6):

$$Y^{\wedge}_E := \operatorname*{holim}_s Y_s/Y.$$

Since $Y^{\wedge}_E$ is the homotopy limit of a tower under Y, it comes equipped with a canonical map $Y \to Y^{\wedge}_E$.

Remark 7.10. In [6], the E-nilpotent completion of Y is denoted "$E^{\wedge}Y$", while the notation "$Y^{\wedge}_E$" we use here is standard in the modern literature.

It turns out that applying $[X, -]_*$ to this tower under Y yields an exact couple, the associated spectral sequence of which is precisely the E-Adams spectral sequence for $[X, Y]_*$.

Proposition 7.11. *Consider the tower under Y constructed in Definition 7.9:*

$$\begin{tikzcd}
Y \arrow[r] & \cdots \arrow[r] & Y/Y_3 \arrow[r, "q"] \arrow[d, dashed, "r"] & Y/Y_2 \arrow[r, "q"] \arrow[d, dashed, "r"] \arrow[dl, leftarrow, "p"'] & Y/Y_1 \arrow[r, "q"] \arrow[d, dashed, "r"] \arrow[dl, leftarrow, "p"'] & Y/Y_0 \arrow[r, equal] \arrow[d, dashed, "r"] \arrow[dl, leftarrow, "p"'] & 0 \\
& & W_3 & W_2 & W_1 & W_0 &
\end{tikzcd}$$

We may extend it to the right by defining $Y/Y_s = W_s = 0$ for $s < 0$. Then by Proposition 2.10, we may apply the functor $[X,-]_*$ which yields the following A-graded unrolled exact couple:

$$\begin{tikzcd}
\cdots \arrow[r] & {[X,Y/Y_{s+2}]_*} \arrow[rr, "q"] \arrow[d, "\delta"] & & {[X,Y/Y_{s+1}]_*} \arrow[rr, "q"] \arrow[d, "\delta"] & & {[X,Y/Y_s]_*} \arrow[rr, "q"] \arrow[d, "\delta"] & & {[X,Y/Y_{s-1}]_*} \arrow[r] \arrow[d, "\delta"] & \cdots \\
& {[X,W_{s+2}]_*} & & {[X,W_{s+1}]_*} \arrow[ull, "p"] & & {[X,W_s]_*} \arrow[ull, "p"] & & {[X,W_{s-1}]_*} \arrow[ull, "p"] &
\end{tikzcd}$$

Thus by [5, §0], there is an induced spectral sequence. This spectral sequence is precisely the E-Adams spectral sequence for $[X,Y]_*$ (Definition 7.2).

Proof. Let $(E'_r(X,Y), d'_r)$ denote this new spectral sequence. For $s \geq 0$, define

$$f_s : [X,Y/Y_s]_* \xrightarrow{c_*} [X,\Sigma Y_s]_* \xrightarrow{(\nu_{Y_s})_*} [X,\Sigma^{\mathbf{1}} Y_s]_* \xrightarrow{s^{\mathbf{1}}_{X,Y_s}} [X,Y_s]_{*-\mathbf{1}},$$

and for $s < 0$ let it be the unique map

$$f_s : [X,Y/Y_s]_* = 0 \to [X,Y_s]_{*-\mathbf{1}} = [X,Y]_{*-1}.$$

For $s \in \mathbb{Z}$, let

$$g_s := \mathrm{id}_{W_s} : [X,W_s]_* \to [X,W_s]_*.$$

We claim these maps $(f_s,g_s)_s$ define a homomorphism of A-graded unrolled exact couples between the unrolled exact couple given above determined by the quotient tower $\{Y/Y_s\}$ under Y, and that obtained by applying $[X,-]_*$ to the canonical E-Adams resolution, i.e., that the following diagram commutes for all $s \in \mathbb{Z}$:

$$\begin{tikzcd}
{[X,Y/Y_s]_*} \arrow[r] \arrow[d, "f_s"'] & {[X,Y/Y_{s-1}]_*} \arrow[r] \arrow[d, dashed, "f_{s-1}"'] & {[X,W_{s-1}]_{*-\mathbf{1}}} \arrow[r] \arrow[d, equal] & {[X,Y/Y_s]_{*-\mathbf{1}}} \arrow[d, "f_s"] \\
{[X,Y_s]_{*-\mathbf{1}}} \arrow[r] & {[X,Y_{s-1}]_{*-\mathbf{1}}} \arrow[r] & {[X,W_{s-1}]_{*-\mathbf{1}}} \arrow[r] & {[X,Y_s]_{*-\mathbf{2}}}
\end{tikzcd}$$

In the case $s \leq 0$, we know $Y/Y_s = Y/Y_{s-1} = W_{s-1} = 0$, so that the top row is entirely 0, and thus the diagram must commute. In the case $s > 0$, by unravelling definitions we have that the diagram becomes

$$\begin{tikzcd}
{[X,Y/Y_s]_*} \arrow[r, "q_*"] \arrow[d, "c_*"'] & {[X,Y/Y_{s-1}]_*} \arrow[rr, "\delta"] \arrow[d, "c_*"] \arrow[dr, "r_*"] & & {[X,W_{s-1}]_{*-\mathbf{1}}} \arrow[r, "p_*"] \arrow[ddd, equal] & {[X,Y/Y_s]_{*-\mathbf{1}}} \arrow[d, "c_*"] \\
{[X,\Sigma Y_s]_*} \arrow[r, "\Sigma i_*"] \arrow[d, "(\nu_{Y_s})_*"'] & {[X,\Sigma Y_{s-1}]_*} \arrow[r, "\Sigma j_*"] \arrow[d, "(\nu_{Y_{s-1}})_*"] & {[X,\Sigma W_{s-1}]_*} \arrow[d, "(\nu_{W_{s-1}})_*"] & & {[X,\Sigma Y_s]_{*-\mathbf{1}}} \arrow[d, "(\nu_{Y_s})_*"] \\
{[X,\Sigma^{\mathbf{1}} Y_s]_*} \arrow[r, "\Sigma^{\mathbf{1}} i_*"] \arrow[d, "s^{\mathbf{1}}_{X,Y_s}"'] & {[X,\Sigma^{\mathbf{1}} Y_{s-1}]_*} \arrow[r, "\Sigma^{\mathbf{1}} j_*"] \arrow[d, "s^{\mathbf{1}}_{X,Y_{s-1}}"] & {[X,\Sigma^{\mathbf{1}} W_{s-1}]_*} \arrow[dr, "s^{\mathbf{1}}_{X,W_{s-1}}"] & & {[X,\Sigma^{\mathbf{1}} Y_s]_{*-\mathbf{1}}} \arrow[d, "s^{\mathbf{1}}_{X,Y_s}"] \\
{[X,Y_s]_{*-\mathbf{1}}} \arrow[r, "i_*"] & {[X,Y_{s-1}]_{*-\mathbf{1}}} \arrow[rr, "j_*"] & & {[X,W_{s-1}]_{*-\mathbf{1}}} \arrow[r, "\partial_*"] \arrow[uuur, "k_*"] & {[X,Y_s]_{*-\mathbf{2}}}
\end{tikzcd}$$

Clearly commutativity of this diagram yields that the given collection of maps define a homomorphism of A-graded unrolled exact couples. Each rectangular region commutes by naturality, as does the middle bottom trapezoidal region. The two regions involving δ and ∂ commute by unravelling how the differential is defined in Proposition 2.10. Finally, the remaining two regions commute by commutativity of Equation 9.

Thus, we have defined a homomorphism of A-graded unrolled exact couples, and it is straightforward to check that therefore the maps g_s lift to well-defined graded homomorphisms $\widetilde{g}^s_r$:

$E_r^s(X,Y) \to E_r^{s\prime}(X,Y)$ for $s \geq 0$ sending a class $[x] \in Z_r^s/B_r^s = E_r^s(X,Y)$ to the class $\widetilde{g}_r^s([x]) := [g_s(x)]$ in $Z_r^{s\prime}/B_r^{s\prime} = E_r^{s\prime}(X,Y)$, which make the following diagrams commute for all $r \geq 1$:

$$\begin{tikzcd} E_r(X,Y) \arrow[r, "\widetilde{g}_r"] \arrow[d, "d_r"'] & E'_r(X,Y) \arrow[d, "d'_r"] \\ E_r(X,Y) \arrow[r, "\widetilde{g}_r"] & E'_r(X,Y) \end{tikzcd} \qquad \begin{tikzcd} \ker d_r \arrow[r, "\widetilde{g}_r"] \arrow[d, two heads] & \ker d'_r \arrow[d, two heads] \\ E_{r+1}(X,Y) \arrow[r, "\widetilde{g}_{r+1}"] & E'_{r+1}(X,Y) \end{tikzcd}$$

(commutativity of the first diagram implies the top arrow in the second diagram is well-defined). Yet we know that each g_s is the identity, so that we shown that $(E_r(X,Y), d_r) = (E'_r(X,Y), d'_r)$, as desired. $\square$

By means of this new presentation of the spectral sequence, we may consider the sense in which the spectral sequence converges to the *limit* $\lim_s [X, Y/Y_s]_*$ of the tower $\{Y/Y_s\}_s$ under Y, by means [5, Theorem 7.4]. First of all, it is standard that since $Y_E^\wedge$ is the homotopy limit of this tower, we have a *Milnor short exact sequence*

$$0 \to \operatorname*{Rlim}_s [X, Y/Y_s]_{*+1} \to [X, Y_E^\wedge]_* \to \lim_s [X, Y/Y_s]_* \to 0$$

(the same argument given in [5, Theorem 4.9] works, although we warn the reader that Boardman has a sign error there — he writes the first term in the short exact sequence with -1, when it should be $+1$). Thus, if $\mathrm{Rlim}_s [X, Y/Y_s]_*$ vanishes, we get an identification of the limit

$$\lim_s [X, Y/Y_s]_* = [X, Y_E^\wedge]_*.$$

By [5, Theorem 7.4], this is further satisfied if the derived E_∞-term $RE_\infty(X,Y)$ is zero, in which case the spectral sequence converges strongly to the limit $[X, Y_E^\wedge]_*$, meaning in particular the natural maps

$$[X, Y_E^\wedge]_* \to \lim_s [X, Y_E^\wedge]_*/F^s[X, Y_E^\wedge]_*$$

$$F^s[X, Y_E^\wedge]_*/F^{s+1}[X, Y_E^\wedge]_* \to E_\infty^{s,*}(X,Y)$$

are isomorphisms, where here F^s is the decreasing filtration on $[X, Y_E^\wedge]_*$ given by

$$F^s[X, Y_E^\wedge]_* := \ker([X, Y_E^\wedge]_* = \lim_s [X, Y/Y_s]_* \to [X, Y/Y_s]_*).$$

8. Future directions

In this section, we briefly touch on some future directions in which one could carry on this work in our general setting.

- One could weaken the cellularity conditions required for the characterization of the E_2 page of the E-Adams spectral sequence for $[X,Y]_*$ (Theorem 7.4) by instead proving a version of Theorem 5.1 for E-*cellular* objects, in the sense of the definition given on pg. 21 in the paper [9].
- One could set up a cohomological version of the E-Adams spectral sequence in $\mathcal{SH}$, as in [9, §3]. In order to show that it agrees with the homological E-Adams spectral sequence we have constructed, one would need to develop some sort anaologue in $\mathcal{SH}$ of the finiteness condition given in [9, Definitions 2.11 & 2.12].
- Much more could be said about properties of the Adams spectral sequence in $\mathcal{SH}$, particularly convergence. Are there connectivity conditions we can place on $\pi_*(Y)$ which guarantee convergence of the E-Adams spectral sequence for $[X,Y]_*$ to $[X, Y_E^\wedge]_*$?
- Under what conditions does a morphisms in $\mathcal{SH}$ induce a homomorphism of E-Adams spectral sequences, as in [28, Proposition 11.4.1]? Similarly, how should one define an E-Adams resolution in $\mathcal{SH}$ in a way such that the E-Adams spectral sequence is independent of the choice of resolution?

- One could define products on the E-Adams spectral sequence in $\mathcal{SH}$ as in [28, Sections 11.7 & 11.8].
- Given a prime p, one could define the "mod-p Moore object" in $\mathcal{SH}$ to be the cofiber of the multiplication-by-p map $S \xrightarrow{p} S$. Using this object, one could define the p-completion $Y_p^\wedge$ of an object Y in $\mathcal{SH}$, as in [28, Definition 11.5.4]. Under which conditions is the canonical map $Y \to Y_p^\wedge$ an isomorphism in $\mathcal{SH}$, and how are $\pi_*(Y)$ and and $\pi_*(Y_p^\wedge)$ related?
- There is a symmetric monoidal realization functor from the motivic stable homotopy category $\mathbf{SH}_{\mathbb{C}}$ over $\operatorname{Spec}\mathbb{C}$ to the classical stable homotopy category **hoSp** given by Betti realization. Under the standard grading anticommutativity conventions for these categories (see Examples 4.8 and 4.9), it further induces a ring homomorphism from the motivic stable homotopy ring (which is $\mathbb{Z}^2$-graded) to the classical stable homotopy ring (which is $\mathbb{Z}$-graded) which sends homogeneous elements of degree (p, q) to elements of degree p. Furthermore, this functor induces a homomorphism of spectral sequence from the $\mathbb{C}$-motivic Adams spectral sequence to the classical Adams spectral sequence. We refer the reader to the Levine's paper [14] for a more in-depth exposition of this story. Similarly, there are realization functors from $\mathbf{SH}_{\mathbb{C}}$ to the C_2-equivariant stable homotopy category with similar properties (see [10, Remarks 3 & 4]). This motivates the idea of a "homomorphism of tensor-triangulated categories with sub-Picard grading", which should be functors which are in some sense compatible with the structure described in Definition 2.3. Furthermore, these functors should induce homomorphisms of homotopy groups and of Adams spectral sequences.
- More work could be done to examine the graded anticommutativity properties of homotopy rings in the G-equvariant stable homotopy category, using the language and methods of Section 4 and [10].
- In the classical, equivariant, and motivic stable homotopy categories, given an abelian group G, there is an associated Eilenberg-MacLane spectrum HG, and this assignment yields a monoidal functor from abelian groups to the stable homotopy category. Given a monoidal functor $\mathbf{Ab} \to \mathcal{SH}$ sending G to HG, what can we say about the HG-Adams spectral sequence? when is the $H\mathbb{F}_p$-completion.
- In the classical, equivariant, and motivic stable homotopy categories, the grading comes from an abelian group A which also happens to be a ring ($A = \mathbb{Z}, \mathbb{Z}^2, RO(G)$). Furthermore, in the classical and motivic stable homotopy categories, this additional ring structure comes into play with regards to the graded anticommutativity properties of $\pi_*(S)$ (e.g., in the classical case, we have a commutativity formula on $\pi_*(S)$ given by $x \cdot y = y \cdot x \cdot (-1)^{|x| \cdot |y|}$). What can be said about a tensor triangulated category with sub-Picard grading coming from a ring, and can we impose any additional conditions on the sub-Picard grading which allows us to say something about the graded anticommutativity properties of $\pi_*(S)$?
- What conditions can we impose on a flat, cellular commtative monoid object (E, μ, e) in $\mathcal{SH}$ which give us a filtration that allows us to create a May spectral sequence for computing the E_2 page of the E-Adams spectral sequence?

As the above exhibits, we have really only scratched the surface of what is possible in this setting, and much more could be done to develop the theory of tensor triangulated categories with sub-Picard grading.

Appendix A. Triangulated categories

A.1. **Triangulated categories and their basic properties.** In this appendix, we fix a triangulated category $(\mathcal{C}, \Sigma, \mathcal{D})$ (Definition 2.1). We will denote the hom-group $\mathcal{C}(X, Y)$ by $[X, Y]$. To start, recall the following definition:

Definition A.1. A sequence

$$X_1 \to X_2 \to \cdots \to X_n$$

of arrows in $\mathcal{C}$ is *exact* if, for any object A in $\mathcal{C}$, the induced sequences

$$[A, X_1] \to [A, X_2] \to \cdots \to [A, X_{n-1}] \to [A, X_n]$$

and

$$[X_n, A] \to [X_{n-1}, A] \to \cdots \to [X_2, A] \to [X_1, A]$$

are exact sequences of abelian groups.

It is straightforward to verify that if we have an exact sequence in $\mathcal{C}$

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \to \cdots \to X_n,$$

then the sequence remains exact if we change the signs of any of the maps involved. We will use this fact often without comment.

Proposition A.2. *Every distinguished triangle is an exact sequence (in the sense of Definition A.1).*

Proof. Suppose we have some distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X.$$

Then first we would like to show that given any object A in $\mathcal{C}$, the sequence

$$[A, X] \xrightarrow{f_*} [A, Y] \xrightarrow{g_*} [A, Z] \xrightarrow{h_*} [A, \Sigma X]$$

is exact. First we show exactness at $[A, Y]$. To see $\operatorname{im} f_* \subseteq \ker g_*$, note it suffices to show that $g \circ f = 0$. Indeed, consider the commuting diagram

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] & \Sigma X \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X
\end{tikzcd}$$

The top row is distinguished by axiom TR1. Thus by TR3, the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X
\end{tikzcd}$$

In particular, commutativity of the second square tells us that $g \circ f = 0$, as desired. Conversely, we'd like to show that $\ker g_* \subseteq \operatorname{im} f_*$. Let $\psi : A \to Y$ be in the kernel of g_*, so that $g \circ \psi = 0$. Consider the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\psi"'] & 0 \arrow[r] \arrow[d] & \Sigma A \arrow[r, "-\Sigma \mathrm{id}_A"] & \Sigma A \\
Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \arrow[r, "-\Sigma f"] & \Sigma Y
\end{tikzcd}$$

The top row is distinguished by axioms TR1 and TR4. The bottom row is distinguished by axiom TR4. Thus by axiom TR3 there exists a map $\widetilde{\phi} : \Sigma A \to \Sigma X$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\psi"'] & 0 \arrow[r] \arrow[d] & \Sigma A \arrow[r, "-\Sigma \mathrm{id}_A"] \arrow[d, "\widetilde{\phi}"'] & \Sigma A \arrow[d, "\Sigma\psi"'] \\ Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \arrow[r, "-\Sigma f"] & \Sigma Y \end{tikzcd}$$

Now, since Σ is an equivalence, it is a full functor, so that in particular there exists some $\phi : A \to X$ such that $\widetilde{\phi} = \Sigma\phi$. Then by faithfullness, we may pull back the right square to get a commuting diagram

$$\begin{tikzcd} A \arrow[r, "-\mathrm{id}_A"] \arrow[d, "\phi"'] & A \arrow[d, "\psi"] \\ X \arrow[r, "-f"'] & Y \end{tikzcd}$$

Hence,

$$f_*(\phi) = f \circ \phi \overset{(*)}{=} -((-f) \circ \phi) = -(\psi \circ (-\mathrm{id}_A)) \overset{(*)}{=} \psi \circ \mathrm{id}_A = \psi,$$

where the equalities marked $(*)$ follow by bilinearity of composition in an additive category. Thus $\psi \in \operatorname{im} f_*$, as desired, meaning $\ker g_* \subseteq \operatorname{im} f_*$.

Now, we have shown that

$$[A, X] \xrightarrow{f_*} [A, Y] \xrightarrow{g_*} [A, Z] \xrightarrow{h_*} [A, \Sigma X]$$

is exact at $[A, Y]$. It remains to show exactness at $[A, Z]$. Yet this follows by the exact same argument given above applied to the sequence obtained from the shifted triangle (TR4)

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

On the other hand, we would like to show that

$$[\Sigma X, A] \xrightarrow{h^*} [Z, A] \xrightarrow{g^*} [Y, A] \xrightarrow{f^*} [X, A]$$

is exact. As above, since we can shift the triangle, it suffices to show exactness at $[Z, A]$. First, since we have shown $g \circ f = 0$, we have $f^* \circ g^* = (g \circ f)^* = 0$, so that $\operatorname{im} g^* \subseteq \ker f^*$, as desired. Conversely, in order to see $\ker f^* \subseteq \operatorname{im} g^*$, suppose $\psi : Y \to A$ is in the kernel of f^*, so that $\psi \circ f = 0$. Consider the following commuting diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "g"] \arrow[d, "\psi"] & Z \arrow[r, "h"] & \Sigma X \\ 0 \arrow[r] & A \arrow[r, equal] & A \arrow[r] & 0 \end{tikzcd}$$

The top row is a distinguished triangle by assumption, and the bottom row is distinguished by axioms TR1 and TR4 for a triangulated category, along with the fact that $\Sigma 0 = 0$ since Σ is additive. Thus by axiom TR3 there exists a map $\phi : Z \to A$ such that $\phi \circ g = \psi$, i.e., $g^*(\phi) = \psi$, so that $\phi \in \operatorname{im} g^*$ as desired. $\square$

Lemma A.3. *Suppose we have a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "j"] & Y \arrow[r, "g"] \arrow[d, "k"] & Z \arrow[r, "h"] \arrow[d, "\ell"] & \Sigma X \arrow[d, "\Sigma j"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & \Sigma X' \end{tikzcd}$$

with both rows distinguished. Then if any two of the maps j, k, and ℓ are isomorphisms, then so is the third.

Proof. Suppose we are given any object W in $\mathcal{C}$, and consider the commutative diagram

$$\begin{tikzcd}
{[W,X]} \arrow[r, "f_*"] \arrow[d, "j_*"] & {[W,Y]} \arrow[r, "g_*"] \arrow[d, "k_*"] & {[W,Z]} \arrow[r, "k_*"] \arrow[d, "\ell_*"] & {[W,\Sigma X]} \arrow[r, "-\Sigma f_*"] \arrow[d, "\Sigma j_*"] & {[W,\Sigma Y]} \arrow[r, "-\Sigma g_*"] \arrow[d, "\Sigma k_*"] & {[W,\Sigma Z]} \arrow[r, "-\Sigma h_*"] \arrow[d, "\Sigma \ell_*"] & {[W,\Sigma^2 X]} \arrow[d, "\Sigma^2 j_*"] \\
{[W,X']} \arrow[r, "f'_*"] & {[W,Y']} \arrow[r, "g'_*"] & {[W,Z']} \arrow[r, "h'_*"] & {[W,\Sigma X']} \arrow[r, "-\Sigma f'_*"] & {[W,\Sigma Y']} \arrow[r, "-\Sigma g'_*"] & {[W,\Sigma Z']} \arrow[r, "-\Sigma h'_*"] & {[W,\Sigma^2 X']}
\end{tikzcd}$$

The rows are exact by Proposition A.2 and repeated applications of axiom TR4. It follows by the five lemma and faithfulness of Σ that if j and k are isomorphisms, then ℓ_* is an isomorphism. Similarly, if k and ℓ are isomorphisms then Σj_* is an isomorphism. Finally, if ℓ and j are isomorphisms, then Σk_* is an isomorphism. The desired result follows by faithfullness of Σ and the Yoneda embedding. □

Proposition A.4. *Given an arrow $f : X \to Y$ in $\mathcal{C}$, there exists an object F_f called the* fiber *of f, and a distinguished triangle*

$$F_f \to X \xrightarrow{f} Y \to \Sigma F_f (\cong C_f).$$

Proof. Since Σ is an equivalence, there exists some functor $\Omega : \mathcal{C} \to \mathcal{C}$ and natural isomorphisms $\varepsilon : \Omega\Sigma \Rightarrow \mathrm{Id}_{\mathcal{C}}$ and $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow \Sigma\Omega$. By axiom TR2, we have a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} C_f \xrightarrow{h} \Sigma X.$$

Now, consider the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] & C_f \arrow[r, "h"] \arrow[d, "\eta_{C_f}"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "f"] & Y \arrow[r, "\widetilde{g}"] & \Sigma\Omega C_f \arrow[r, "\widetilde{h}"] & \Sigma X
\end{tikzcd}$$

where $\widetilde{g} = \eta_{C_f} \circ g$, and $\widetilde{h} = h \circ \eta_{C_f}^{-1}$. Since each vertical map is an isomorphism and the top row is distinguished, the bottom row is also distinguished by axiom TR0. Now, since Σ is an equivalence of categories, it is faithful, so that in particular there exists some map $k : \Omega C_f \to X$ such that $\Sigma k = -\widetilde{h} \implies -\Sigma k = \widetilde{h}$. Thus, we have a distinguished triangle of the form

$$X \xrightarrow{f} Y \xrightarrow{\widetilde{g}} \Sigma\Omega C_f \xrightarrow{-\Sigma k} \Sigma X.$$

Finally, by axiom TR4, we get a distinguished triangle

$$\Omega C_f \xrightarrow{k} X \xrightarrow{f} Y \xrightarrow{\widetilde{g}} \Sigma\Omega C_f,$$

so we may define the fiber of f to be ΩC_f. □

A.2. **Homotopy (co)limits in a triangulated category.** In this subsection, we will assume $\mathcal{C}$ has countable products and coproducts.

Definition A.5 ([19, Definition 1.6.4])**.** Let

$$X_0 \xrightarrow{j_1} X_1 \xrightarrow{j_2} X_2 \xrightarrow{j_3} X_3 \to \cdots$$

be a sequence of objects and morphisms in $\mathcal{C}$. The *homotopy colimit* of the sequence, denoted $\operatorname{hocolim} X_i$, is given (up to non-canonical isomorphism) as the cofiber of the map

$$\bigoplus_{i=0}^{\infty} X_i \xrightarrow{1-\text{shift}} \bigoplus_{i=0}^{\infty} X_i,$$

where the shift map $\bigoplus_{i=0}^{\infty} X_i \xrightarrow{\text{shift}} \bigoplus_{i=0}^{\infty} X_i$ is understood to be the direct sum of $j_{i+1} : X_i \to X_{i+1}$, i.e., by the universal property of the coproduct, it is induced by the maps

$$X_s \xrightarrow{j_{s+1}} X_{s+1} \hookrightarrow \bigoplus_{i=0}^{\infty} X_i.$$

Definition A.6. Assume that $\mathcal{C}$ has countable products, and let

$$\cdots \to X_3 \xrightarrow{j_3} X_2 \xrightarrow{j_2} X_1 \xrightarrow{j_1} X_0$$

be a sequence of objects and morphisms in $\mathcal{C}$. The *homotopy limit* of the sequence, denoted $\operatorname{holim} X_i$, is given (up to non-canonical isomorphism) as the fiber (Proposition A.4) of the map

$$\prod_{i=0}^{\infty} X_i \xrightarrow{1-\text{shift}} \prod_{i=0}^{\infty},$$

where the shift map $\prod_{i=0}^{\infty} X_i \xrightarrow{\text{shift}} \prod_{i=0}^{\infty} X_i$ is understood to be the product of $j_i : X_i \to X_{i-1}$, i.e., by the universal property of the product, it is induced by the maps

$$\prod_{i=0}^{\infty} X_i \twoheadrightarrow X_{s+1} \xrightarrow{j_{s+1}} X_s.$$

A.3. **Adjointly triangulated categories.** From now on, we will assume that $\mathcal{C}$ is an *adjointly* triangulated category (Definition 2.8) with inverse shift Ω, unit $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow \Sigma\Omega$, and counit $\varepsilon : \Omega\Sigma \Rightarrow \mathrm{Id}_{\mathcal{C}}$.

Lemma A.7. *Given a triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

it can be shifted to the left to obtain a distinguished triangle

$$\Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{f} Y \xrightarrow{\widetilde{\Omega g}} \Sigma\Omega Z,$$

where $\widetilde{h} : \Omega Z \to X$ is the adjoint of $h : Z \to \Sigma X$ and $\widetilde{\Omega g} : Y \to \Sigma\Omega Z$ is the adjoint of $\Omega g : \Omega Y \to \Omega Z$.

Proof. Note that unravelling definitions, $\widetilde{h}$ and $\widetilde{g}$ are the compositions

$$\widetilde{h} : \Omega Z \xrightarrow{\Omega h} \Omega\Sigma X \xrightarrow{\varepsilon_X} X \qquad \text{and} \qquad \widetilde{\Omega g} : Y \xrightarrow{\eta_Y} \Sigma\Omega Y \xrightarrow{\Sigma\Omega g} \Sigma\Omega Z.$$

Now consider the following diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] & Z \arrow[r, "h"] \arrow[d, "\eta_Z"'] & \Sigma X \arrow[d, equal] \\ X \arrow[r, "f"] & Y \arrow[r, "\widetilde{\Omega g}"] & \Sigma\Omega Z \arrow[r, "\Sigma\widetilde{h}"] & \Sigma X \end{tikzcd} \tag{10}$$

The left square commutes by definition. To see that the middle square commutes, expanding definitions, note it is given by the following diagram:

$$\begin{tikzcd} Y \arrow[rr, "g"] \arrow[d, equal] & & Z \arrow[d, "\eta_Y"] \\ Y \arrow[r, "\eta_Y"] & \Sigma\Omega Y \arrow[r, "\Sigma\Omega g"] & \Sigma\Omega Z \end{tikzcd}$$

and this commutes by naturality of η. To see that the right square commutes, consider the following diagram:

$$\begin{tikzcd} Z \arrow[rr, "h"] \arrow[d, "\eta_Z"'] & & \Sigma X \arrow[dl, "\eta_{\Sigma X}"'] \arrow[d, equal] \\ \Sigma\Omega Z \arrow[r, "\Sigma\Omega h"'] & \Sigma\Omega\Sigma X \arrow[r, "\Sigma\varepsilon_X"'] & \Sigma X \end{tikzcd}$$

By functoriality of Σ, the bottom composition is $\Sigma\widetilde{h}$. The left region commutes by naturality of η. Commutativity of the right region is precisely one of the the zig-zag identities. Hence, since diagram (10) commutes, the vertical arrows are isomorphisms, and the top row is distinguished, we have that the bottom row is distinguished as well by axiom TR0. Then by axiom TR4, since $(f, \widetilde{\Omega g}, \Sigma\widetilde{h})$ is distinguished, so is the triangle

$$\Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{f} Y \xrightarrow{\widetilde{\Omega g}} \Sigma\Omega Z. \qquad \square$$

Lemma A.8. *Given a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

for any $n > 0$, the triangle

$$\Omega^n X \xrightarrow{(-1)^n\Omega^n f} \Omega^n Y \xrightarrow{(-1)^n\Omega^n g} \Omega^n Z \xrightarrow{(-1)^n\Omega^n h} \Omega^n\Sigma X \cong \Sigma\Omega^n X,$$

is distinguished, where the final isomorphism is given by the composition

$$\Omega^n\Sigma X = \Omega^{n-1}\Omega\Sigma X \xrightarrow{\Omega^{n-1}\varepsilon_X} \Omega^{n-1}X \xrightarrow{\eta_{\Omega^{n-1}X}} \Sigma\Omega\Omega^{n-1}X = \Sigma\Omega^n X.$$

Proof. We give a proof by induction. First we show the case $n = 1$. Note by Lemma A.7, we have that given a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

we can shift it to the left to obtain a distinguished triangle

$$\Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{f} Y \xrightarrow{\widetilde{\Omega g}} \Sigma\Omega Z,$$

where $\widetilde{h}$ is the adjoint of $h : Z \to \Sigma X$ and $\widetilde{\Omega g}$ is the adjoint of $\Omega g : \Omega Y \to \Omega Z$. If we apply this shifting operation again, we get the distinguished triangle

$$\Omega Y \xrightarrow{-\widetilde{\widetilde{\Omega g}}} \Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{\widetilde{\Omega f}} \Sigma\Omega Y,$$

where unravelling definitions, $\widetilde{\Omega f}$ is the right adjoint of $\Omega f : \Omega X \to \Omega Y$ and $\widetilde{\widetilde{\Omega g}}$ is the right adjoint of $\widetilde{\Omega g}$, which itself is the left adjoint of Ωg, so $\widetilde{\widetilde{\Omega g}} = \Omega g$. Hence we have a distinguished triangle

$$\Omega Y \xrightarrow{-\Omega g} \Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{\widetilde{\Omega f}} \Sigma\Omega Y.$$

We may again shift this triangle again and the above arguments yield the distinguished triangle

$$\Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{-\Omega g} \Omega Z \xrightarrow{\widetilde{\Omega(-\widetilde{h})}} \Sigma\Omega X,$$

where $\widetilde{\Omega(-\widetilde{h})}$ is the right adjoint of $\Omega(-\widetilde{h}) = -\Omega\widetilde{h} : \Omega\Omega Z \to \Omega X$. Explicitly unravelling definitions, $\widetilde{\Omega(-\widetilde{h})} = \widetilde{-\Omega\widetilde{h}}$ is the composition

$$\begin{aligned}
[\Omega Z \xrightarrow{\eta_{\Omega Z}} \Sigma\Omega\Omega Z \xrightarrow{\Sigma(-\Omega\widetilde{h})} \Sigma\Omega X] &= -[\Omega Z \xrightarrow{\eta_{\Omega Z}} \Sigma\Omega\Omega Z \xrightarrow{\Sigma\Omega\widetilde{h}} \Sigma\Omega X] \\
&= -[\Omega Z \xrightarrow{\eta_{\Omega Z}} \Sigma\Omega\Omega Z \xrightarrow{\Sigma\Omega\Omega h} \Sigma\Omega\Omega\Sigma X \xrightarrow{\Sigma\Omega\varepsilon_X} \Sigma\Omega X] \\
&= -[\Omega Z \xrightarrow{\Omega h} \Omega\Sigma X \xrightarrow{\varepsilon_X} X \xrightarrow{\eta_X} \Sigma\Omega X],
\end{aligned}$$

where the first equality follows by additivity of Σ and additivity of composition, the second follows by further unravelling how $\widetilde{h}$ is defined, and the third follows by naturality of η, which tells us the following diagram commutes:

$$\begin{tikzcd}
\Omega Z \arrow[r, "\Omega h"] \arrow[d, "\eta_{\Omega Z}"] & \Omega\Sigma X \arrow[r, "\varepsilon_X"] \arrow[d, "\eta_{\Omega\Sigma X}"] & X \arrow[d, "\eta_X"] \\
\Sigma\Omega\Omega Z \arrow[r, "\Sigma\Omega\Omega h"'] & \Sigma\Omega\Omega\Sigma X \arrow[r, "\Sigma\Omega\varepsilon_X"'] & \Sigma\Omega X
\end{tikzcd}$$

Thus indeed we have a distinguished triangle

$$\Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{-\Omega g} \Omega Z \xrightarrow{-\Omega h} \Omega\Sigma X \cong \Sigma\Omega X,$$

where the last isomorphism is $\eta_X \circ \varepsilon_X$, as desired.

Now, we show the inductive step. Suppose we know that given a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

that for some $n > 0$ the triangle

$$\Omega^n X \xrightarrow{(-1)^n\Omega^n f} \Omega^n Y \xrightarrow{(-1)^n\Omega^n g} \Omega^n Z \xrightarrow{(-1)^n h^n} \Sigma\Omega^n X,$$

is distinguished, where $h^n : \Omega^n Z \to \Sigma\Omega^n X$ is the composition

$$\Omega^n Z \xrightarrow{\Omega^n h} \Omega^n\Sigma X \xrightarrow{\Omega^{n-1}\varepsilon_X} \Omega^{n-1} X \xrightarrow{\eta_{\Omega^{n-1}X}} \Sigma\Omega^n X.$$

Then by applying the $n = 1$ case to this triangle, we get that the following triangle is distinguished

$$\Omega^{n+1} X \xrightarrow{-\Omega((-1)^n\Omega^n f)} \Omega^{n+1} Y \xrightarrow{-\Omega((-1)^n\Omega^n g)} \Omega^{n+1} Z \xrightarrow{-\Omega((-1)^n h^n)} \Omega\Sigma\Omega^n X \cong \Sigma\Omega^{n+1} X,$$

where the final isomorphism is the composition

$$\Omega\Sigma\Omega^n X \xrightarrow{\varepsilon_{\Omega^n X}} \Omega^n X \xrightarrow{\eta_{\Omega^n X}} \Sigma\Omega\Omega^n X = \Sigma\Omega^{n+1} X.$$

We claim that this is precisely the distinguished triangle given in the statement of the lemma for $n+1$. First of all, note that $-\Omega((-1)^n\Omega^n f) = (-1)^{n+1}\Omega^{n+1} f$, $-\Omega((-1)^n\Omega^n g) = (-1)^{n+1}\Omega^{n+1} g$, and $-\Omega((-1)^n h^n) = (-1)^{n+1}\Omega h^n$ by additivity of Ω, so that the triangle becomes

$$\Omega^{n+1} X \xrightarrow{(-1)^{n+1}\Omega^{n+1} f} \Omega^{n+1} Y \xrightarrow{(-1)^{n+1}\Omega^{n+1} g} \Omega^{n+1} Z \xrightarrow{(-1)^{n+1}\Omega h^n} \Omega\Sigma\Omega^n X \cong \Sigma\Omega^{n+1} X. \tag{11}$$

Thus, in order to prove the desired characterization, it remains to show this diagram commutes:

$$\begin{tikzcd}
\Omega^{n+1} Z \arrow[r, "(-1)^{n+1}\Omega h^n"] \arrow[d, "(-1)^{n+1}\Omega^{n+1} h"'] & \Omega\Sigma\Omega^n X \arrow[r, "\varepsilon_{\Omega^n X}"] & \Omega^n X \arrow[d, "\eta_{\Omega^n X}"] \\
\Omega^{n+1}\Sigma X \arrow[r, "\Omega^n\varepsilon_X"'] & \Omega^n X \arrow[r, "\eta_{\Omega^n X}"'] & \Sigma\Omega^{n+1} X
\end{tikzcd}$$

(The top composition is the last two arrows in diagram (11), and the bottom composition is the last two arrows in the diagram in the statement of the lemma). Unravelling how h^n is constructed, by additivity of Ω it further suffices to show the outside of the following diagram commutes:

$$\begin{tikzcd}
\Omega^{n+1}Z \arrow[r, "(-1)^{n+1}\Omega^{n+1}h"] \arrow[dd, "(-1)^{n+1}\Omega^{n+1}h"'] & \Omega^{n+1}\Sigma X \arrow[r, "\Omega^n\varepsilon_X"] & \Omega^n X \arrow[r, "\Omega\eta_{\Omega^{n-1}X}"] \arrow[dd, equal] & \Omega\Sigma\Omega^n X \arrow[d, "\varepsilon_{\Omega^n X}"] \\
 & & & \Omega^n X \arrow[d, "\eta_{\Omega^n X}"] \\
\Omega^{n+1}\Sigma X \arrow[rr, "\Omega^n\varepsilon_X"'] & & \Omega^n X \arrow[r, "\eta_{\Omega^n X}"'] \arrow[ur, equal] & \Sigma\Omega^{n+1}X
\end{tikzcd}$$

The left rectangle and bottom right triangle commute by definition. Finally, commutativity of the top right trapezoid is precisely one of the zig-zag identities applied to $\Omega^{n-1}X$. Hence, we have shown the desired result. ☐

Proposition A.9. *Given a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

let $\widetilde{h} : \Omega Z \to X$ be the left adjoint of h. Then the following infinite sequence is exact:

$$\begin{tikzcd}
 & & \cdots \arrow[dll] & & \\
\Omega^{n+1}Z \arrow[r, "(-1)^{n+1}\Omega^n\widetilde{h}"'] & \Omega^n X \arrow[r, "(-1)^n\Omega^n f"'] & \Omega^n Y \arrow[r, "(-1)^n\Omega^n g"] & \Omega^n Z \arrow[r, "(-1)^n\Omega^{n-1}\widetilde{h}"] & \Omega^{n-1}X \arrow[dll] \\
 & & \cdots \arrow[dll] & & \\
\Omega Z \arrow[r, "-\widetilde{h}"'] & X \arrow[r, "f"'] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \arrow[dll] \\
 & & \cdots \arrow[dll] & & \\
\Sigma^{n-1}Z \arrow[r, "(-1)^{n-1}\Sigma^n h"'] & \Sigma^n X \arrow[r, "(-1)^n\Sigma^n f"'] & \Sigma^n Y \arrow[r, "(-1)^n\Sigma^n g"] & \Sigma^n Z \arrow[r, "(-1)^n\Sigma^n h"] & \Sigma^{n+1}X \arrow[dll] \\
 & & \cdots & &
\end{tikzcd}$$

In particular, it remains exact even if we remove the signs.

Proof. Exactness of

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is Proposition A.2 and axiom TR4. By induction using axiom TR4, for $n > 0$ we get that each contiguous composition of three maps below is a distinguished triangle:

$$\Sigma^n X \xrightarrow{(-1)^n\Sigma^n f} \Sigma^n Y \xrightarrow{(-1)^n\Sigma^n g} \Sigma^n Z \xrightarrow{(-1)^n\Sigma^n h} \Sigma^{n+1}X \xrightarrow{(-1)^{n+1}\Sigma^{n+1}f} \Sigma^{n+1}Y,$$

thus the sequence is exact by Proposition A.2. It remains to show exactness of the LES to the left of Y. It suffices to show that the row in the following diagram is exact for all $n > 0$:
(12)

$$\begin{tikzcd}
\Omega^n X \arrow[r, "(-1)^n\Omega^n f"] & \Omega^n Y \arrow[r, "(-1)^n\Omega^n g"] & \Omega^n Z \arrow[rr, "(-1)^n\Omega^{n-1}(\varepsilon_X\circ\Omega h)"] \arrow[dr, "(-1)^n\Omega^n h"'] & & \Omega^{n-1}X \arrow[r, "(-1)^{n-1}\Omega^{n-1}f"] & \Omega^{n-1}Y \\
& & & \Omega^n\Sigma X \arrow[ur, "\Omega^{n-1}\varepsilon_X"'] & &
\end{tikzcd}$$

First of all, to see exactness at $\Omega^n Y$ and $\Omega^n Z$, consider the following commutative diagram:

$$\begin{tikzcd}
\Omega^n X \arrow[r, "(-1)^n\Omega^n f"] \arrow[dd, equal] & \Omega^n Y \arrow[r, "(-1)^n\Omega^n g"] \arrow[dd, equal] & \Omega^n Z \arrow[rr, "(-1)^n\Omega^{n-1}(\varepsilon_X\circ\Omega h)"] \arrow[dd, equal] \arrow[dr, "(-1)^n\Omega^n h"'] & & \Omega^{n-1}X \arrow[dd, "\eta_{\Omega^{n-1}X}"] \\
& & & \Omega^n\Sigma X \arrow[ur, "\Omega^{n-1}\varepsilon_X"'] & \\
\Omega^n X \arrow[r, "(-1)^n\Omega^n f"] & \Omega^n Y \arrow[r, "(-1)^n\Omega^n g"] & \Omega^n Z \arrow[rr, dashed] \arrow[ur, "(-1)^n\Omega^n h"] & & \Sigma\Omega^n X
\end{tikzcd}$$

(here the dashed arrow is the morphism which makes the diagram commute). The bottom row is distinguished by Lemma A.8. Then by axiom TR0, the top row is distinguished, and thus exact by Proposition A.2. Thus we have shown exactness of (12) at $\Omega^n Y$ and $\Omega^n Z$. It remains to show exactness at $\Omega^{n-1}X$. In the case $n = 1$, we want to show exactness at X in the following diagram:

$$\begin{tikzcd}
\Omega Z \arrow[rr, "-(\varepsilon_X\circ\Omega h)"] \arrow[dr, "-\Omega h"'] & & X \arrow[r, "f"] & Y \\
& \Omega\Sigma X \arrow[ur, "\varepsilon_X"'] & &
\end{tikzcd}$$

Unravelling definitions, $\varepsilon_X \circ \Omega h$ is precisely the adjoint $\widetilde{h} : \Omega Z \to X$ of $h : Z \to \Sigma X$, in which case we have that the row in the above diagram fits into a distinguished triangle by Lemma A.7, and thus it is exact by Proposition A.2. To see exactness at $\Omega^{n-1}X$ in diagram (12), note that if we apply Lemma A.7 to the sequence Lemma A.8 for $n-1$, then we get that the following composition fits into a distinguished triangle, and is thus exact:

$$\Omega^n Z \xrightarrow{-k} \Omega^{n-1}X \xrightarrow{(-1)^{n-1}\Omega^{n-1}f} \Omega^{n-1}Y,$$

where $k : \Omega(\Omega^{n-1}Z) \to \Omega^{n-1}X$ is the adjoint of the composition

$$\Omega^{n-1}Z \xrightarrow{(-1)^{n-1}\Omega^{n-1}h} \Omega^{n-1}\Sigma X \xrightarrow{\Omega^{n-2}\varepsilon_X} \Omega^{n-2}X \xrightarrow{\eta_{\Omega^{n-2}X}} \Sigma\Omega^{n-1}X.$$

Further expanding how adjoints are constructed, k is the composition

$$\Omega^n Z \xrightarrow{(-1)^{n-1}\Omega^n h} \Omega^n\Sigma X \xrightarrow{\Omega^{n-1}\varepsilon_X} \Omega^{n-1}X \xrightarrow{\Omega\eta_{\Omega^{n-2}X}} \Omega\Sigma\Omega^{n-1}X \xrightarrow{\varepsilon_{\Omega^{n-1}X}} \Omega^{n-1}X.$$

Thus, in order to show exactness of (12) at $\Sigma^{n-1}X$, it suffices to show that $k = (-1)^{n-1}\Omega^{n-1}(\varepsilon_X \circ \Omega h)$. To that end, consider the following diagram:

$$\begin{tikzcd}
\Omega^n Z \arrow[r, "(-1)^{n-1}\Omega^n h"] \arrow[d, "(-1)^{n-1}\Omega^n h"'] & \Omega^n\Sigma X \arrow[r, "\Omega^{n-1}\varepsilon_X"] & \Omega^{n-1}X \arrow[r, "\Omega\eta_{\Omega^{n-2}X}"] \arrow[dr, equal] & \Omega\Sigma\Omega^{n-1}X \arrow[d, "\varepsilon_{\Omega^{n-1}X}"] \\
\Omega^n\Sigma X \arrow[rrr, "\Omega^{n-1}\varepsilon_X"'] & & & \Omega^{n-1}X
\end{tikzcd}$$

The top composition is k, while the bottom composition is $(-1)^{n-1}\Omega^{n-1}(\varepsilon_X \circ \Omega h)$. The left region commutes by definition, while commutativity of the right region is precisely one of the zig-zag identities applied to $\Omega^{n-2}X$. Thus, we have shown that $-k = (-1)^n\Omega^{n-1}(\varepsilon_X \circ \Omega h)$, so (12) is exact at $\Omega^{n-1}X$, as desired. □

A.4. **Tensor triangulated categories.** In what follows, we fix a tensor triangulated category $(\mathcal{C}, \otimes, S, \Sigma, e, \mathcal{D})$ (Definition 2.2).

Lemma A.10. *Let $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} D$ be any sequence isomorphic to a distinguished triangle. Then given any E in $\mathcal{C}$, the sequences*

$$E \otimes A \xrightarrow{E\otimes a} E \otimes B \xrightarrow{E\otimes b} E \otimes C \xrightarrow{E\otimes c} E \otimes D$$

and

$$A \otimes E \xrightarrow{a\otimes E} B \otimes E \xrightarrow{b\otimes E} C \otimes E \xrightarrow{c\otimes E} D \otimes E$$

are exact.

Proof. Since (a, b, c) is isomorphic to a distinguished triangle, there exists a commuting diagram in $\mathcal{SH}$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[r, "g"] \arrow[d, "\beta"'] & Z \arrow[r, "h"] \arrow[d, "\gamma"'] & \Sigma X \arrow[d, "\delta"'] \\
A \arrow[r, "a"'] & B \arrow[r, "b"'] & C \arrow[r, "c"'] & D
\end{tikzcd}$$

where the top row is distinguished and the vertical arrows are isomorphisms. Then the following diagram commutes by functoriality of $-\otimes-$:

$$\begin{tikzcd}
E\otimes X \arrow[r, "E\otimes f"] \arrow[dd, "E\otimes\alpha"'] & E\otimes Y \arrow[r, "E\otimes g"] \arrow[dd, "E\otimes\beta"'] & E\otimes Z \arrow[rr, "E\otimes' h"] \arrow[dd, "E\otimes\gamma"'] \arrow[dr, dashed, "E\otimes h"] & & \Sigma(E\otimes X) \arrow[dd, dashed, "(E\otimes\delta)\circ(e'_{E,X})^{-1}"] \\
 & & & E\otimes\Sigma X \arrow[ur, dashed, "e'_{E,X}"] \arrow[dr, dashed, "E\otimes\delta"] & \\
E\otimes A \arrow[r, "E\otimes a"'] & E\otimes B \arrow[r, "E\otimes b"'] & E\otimes C \arrow[rr, "E\otimes c"'] & & E\otimes D
\end{tikzcd}$$

The top triangle is distinguished by axiom TT3 for a tensor triangulated category, thus exact by Proposition A.2, so that the bottom triangle is also exact since the vertical arrows are isomorphisms and each square commutes. Similarly, the following diagram also commutes by functoriality of $-\otimes-$:

$$\begin{tikzcd}
X\otimes E \arrow[r, "f\otimes E"] \arrow[dd, "\alpha\otimes E"'] & Y\otimes E \arrow[r, "g\otimes E"] \arrow[dd, "\beta\otimes E"'] & Z\otimes E \arrow[rr, "h\otimes' E"] \arrow[dd, "\gamma\otimes E"'] \arrow[dr, dashed, "h\otimes E"] & & \Sigma(X\otimes E) \arrow[dd, "(\delta\otimes E)\circ e^{-1}_{X,E}"] \\
 & & & \Sigma X\otimes E \arrow[ur, dashed, "e_{X,E}"] \arrow[dr, dashed, "\delta\otimes E"] & \\
A\otimes E \arrow[r, "a\otimes E"'] & B\otimes E \arrow[r, "b\otimes E"'] & C\otimes E \arrow[rr, "c\otimes E"'] & & D\otimes E
\end{tikzcd}$$

The top row is distinguished by axiom TT3 for a tensor triangulated category, thus exact by Proposition A.2, so that the bottom triangle is also exact since the vertical arrows are isomorphisms and each square commutes. □

Proposition A.11. *Suppose we have a distinguished triangle*

$$X \to Y \to Z \to \Sigma X$$

in $\mathcal{C}$. Then given any object E in $\mathcal{C}$, the long exact sequence given in Proposition A.9 remains exact after applying $E \otimes -$ or $-\otimes E$.

Proof. Recall that in the proof of Proposition A.9 we showed that the sequence was exact by showing that any two consecutive maps were isomorphic to a part of a distinguished triangle. Then the desired result follows from Lemma A.10. □

Appendix B. A-graded objects

In this appendix, we fix an abelian group A once and for all. We assume the reader is familiar with the basic theory of (non-commutative, unital) rings and modules over them.

B.1. A-graded abelian groups, rings, and modules.

Definition B.1. An *A-graded abelian group* is an abelian group B along with a subgroup $B_a \leq B$ for each $a \in A$ such that the canonical map

$$\bigoplus_{a \in A} B_a \to B$$

sending $(x_a)_{a\in A}$ to $\sum_{a\in A} x_a$ is an isomorphism. Given two A-graded abelian groups B and C, a homomorphism $f : B \to C$ is a *homomorphism of A-graded abelian groups*, or just an *A-graded homomorphism*, if it preserves the grading, i.e., if it restricts to a map $B_a \to C_a$ for all $a \in A$.

We denote the category of A-graded abelian groups and A-graded homomorphisms between them by $\mathbf{Ab}^A$

It is easy to see that an A-graded abelian group B is generated by its *homogeneous* elements, that is, nonzero elements $x \in B$ such that there exists some $a \in A$ with $x \in B_a$. Furthermore, by the universal property of the coproduct, given two A-graded abelian groups B and C, the data of an A-graded homomorphism $\varphi_a : B \to C$ is precisely the data of homomorphisms $\varphi_a : B_a \to C_a$.

Remark B.2. Clearly the condition that the canonical map $\bigoplus_{a\in A} B_a \to B$ is an isomorphism requires that $B_a \cap B_b = 0$ if $a \neq b$. In particular, given a homogeneous element $x \in B$, there exists precisely one $a \in A$ such that $x \in B_a$. We call this a the *degree* of x, and we write $|x| = a$.

Definition B.3. An *A-graded ring* is a ring R such that its underlying abelian group R is A-graded and the multiplication map $R \times R \to R$ restricts to $R_a \times R_b \to R_{a+b}$ for all $a, b \in A$. A morphism of A-graded rings is a ring homomorphism whose underlying homomorphism of abelian groups is A-graded.

Explicitly, given an A-graded ring R and homogeneous elements $x, y \in R$, we must have $|xy| = |x| + |y|$. For example, given some field k, the ring $R = k[x, y]$ is $\mathbb{Z}^2$-graded, where given $(n, m) \in \mathbb{Z}^2$, $R_{n,m}$ is the subgroup of those monomials of the form $ax^n y^m$ for some $a \in k$.

Definition B.4. Let R be an A-graded ring. A *left A-graded R-module* M is a left R-module M such that M is an A-graded abelian group and the action map $R \times M \to M$ restricts to a map $R_a \times M_b \to M_{a+b}$ for all $a, b \in A$. Right A-graded R-modules are defined similarly. Finally, an A-graded R-bimodule is an A-graded abelian group M which has the structure of both an A-graded left and right R-module such that given $r, s \in R$ and $m \in M$, $r \cdot (m \cdot s) = (r \cdot m) \cdot s$.

Morphisms between A-graded R-modules are precisely R-module homomorphisms whose underlying group homomorphisms are A-graded. We write $R\text{-}\mathbf{Mod}^A$ for the category of left A-graded R-modules and $\mathbf{Mod}^A\text{-}R$ for the category of right A-graded R-modules.

Remark B.5. It is straightforward to see that an A-graded abelian group is equivalently an A-graded $\mathbb{Z}$-module, where here we are considering $\mathbb{Z}$ as an A-graded ring concentrated in degree 0. Thus any result below about A-graded modules applies equally to A-graded abelian groups.

Remark B.6. We often will denote an A-graded R-module M by M_*. Given some $a \in A$, we can define the shifted A-graded abelian group M_{*+a} whose b^{th} component is M_{b+a}. We will also sometimes write $\Sigma^a M$ to denote the shifted module M_{*-a}.

Definition B.7. More generally, given two A-graded R-modules M and N and some $d \in A$, an R-module homomorphism $f : M \to N$ is *an A-graded homomorphism of degree d* if it restricts to a map $M_a \to N_{a+d}$ for all $a \in A$. Thus, an A-graded homomorphism of degree d from M to N is equivalently an A-graded homomorphism $M_* \to N_{*+d}$ or an A-graded homomorphism $M_{*-d} \to N$. Given some $a \in A$ and left (resp. right) R-modules M and N, we will write

$$\operatorname{Hom}_R^d(M, N) = \operatorname{Hom}_R(M_*, N_{*+d}) = \operatorname{Hom}_R(M_{*-d}, N_*)$$

to denote the set of A-graded homomorphisms of degree d from M to N, and simply

$$\operatorname{Hom}_R(M, N)$$

to denote the set of degree-0 A-graded homomorphisms from M to N. Clearly A-graded homomorphisms may be added and subtracted, so these are further abelian groups. Thus we have an A-graded abelian group

$$\operatorname{Hom}_R^*(M, N).$$

Unless stated otherwise, an "A-graded homomorphism" will always refer to an A-graded homomorphism of degree 0.

Oftentimes when constructing A-graded rings, we do so only by defining the product of homogeneous elements, like so:

Lemma B.8. *Suppose we have an A-graded abelian group R, a distinguished element $1 \in R_0$, and $\mathbb{Z}$-bilinear maps $m_{a,b} : R_a \times R_b \to R_{a+b}$ for all $a, b \in A$. Further suppose that for all $x \in R_a$, $y \in R_b$, and $z \in R_c$, we have*

$$m_{a+b,c}(m_{a,b}(x, y), z) = m_{a,b+c}(x, m_{b,c}(y, z)) \qquad \text{and} \qquad m_{a,0}(x, 1) = m_{0,a}(1, x) = x.$$

Then there exists a unique multiplication map $m : R \times R \to R$ which endows R with the structure of an A-graded ring and restricts to $m_{a,b}$ for all $a, b \in A$.

Proof. Given $r, s \in R$, since $R \cong \bigoplus_{a\in A} R_a$, we may uniquely decompose r and s into homogeneous elements as $r = \sum_{a\in A} r_a$ and $s = \sum_{a\in A} s_a$ with each $r_a, s_a \in R_a$ such that only finitely many of the r_a's and s_a's are nonzero. Then in order to define a distributive product $R \times R \to R$ which restricts to $m_{a,b} : R_a \times R_b \to R_{a+b}$, note we *must* define

$$r \cdot s = \left(\sum_{a\in A} r_a\right) \cdot \left(\sum_{b\in A} s_b\right) = \sum_{a,b\in A} r_a \cdot s_b = \sum_{a,b\in A} m_{a,b}(r_a, s_b).$$

Thus, we have shown uniqueness. It remains to show this product actually gives R the structure of a ring. First we claim that the sum on the right is actually finite. Note there exists only finitely many nonzero r_a's and s_b's, and if $s_b = 0$ then

$$m_{a,b}(r_a, 0) = m_{a,b}(r_a, 0 + 0) \overset{(*)}{=} m_{a,b}(r_a, 0) + m_{a,b}(r_a, 0) \implies m_{a,b}(r_a, 0) = 0,$$

where $(*)$ follows from bilinearity of $m_{a,b}$. A similar argument yields that $m_{a,b}(0, s_b) = 0$ for all $a, b \in A$. Hence indeed $m_{a,b}(r_a, s_b)$ is zero for all but finitely many pairs $(a, b) \in A^2$, as desired. Observe that in particular

$$(r \cdot s)_a = \sum_{b+c=a} m_{b,c}(r_b, s_c) = \sum_{b\in A} m_{b,a-b}(r_b, s_{a-b}) = \sum_{c\in A} m_{a-c,c}(r_{a-c}, s_c).$$

Now we claim this multiplication is associative. Given $t = \sum_{a\in A} t_a \in R$, we have

$$\begin{aligned}
(r\cdot s)\cdot t &= \sum_{a,b\in A} m_{a,b}((r\cdot s)_a, t_b)\\
&= \sum_{a,b\in A} m_{a,b}\left(\sum_{c\in A} m_{a-c,c}(r_{a-c}, s_c), t_b\right)\\
&\overset{(1)}{=} \sum_{a,b,c\in A} m_{a,b}(m_{a-c,c}(r_{a-c}, s_c), t_b)\\
&\overset{(2)}{=} \sum_{a,b,c\in A} m_{c,a+b-c}(r_c, m_{a-c,b}(s_{a-c}, t_b))\\
&\overset{(3)}{=} \sum_{a,b,c\in A} m_{a,c}(r_a, m_{b,c-b}(s_b, t_{c-b}))\\
&\overset{(1)}{=} \sum_{a,c\in A} m_{a,c}\left(r_a, \sum_{b\in A} m_{b,c-b}(s_b, t_{c-b})\right)\\
&= \sum_{a,c\in A} m_{a,c}(r_a, (s\cdot t)_c) = r\cdot(s\cdot t),
\end{aligned}$$

where each occurrence of (1) follows by bilinearity of the $m_{a,b}$'s, each occurrence of (2) is associativity of the $m_{a,b}$'s, and (3) is obtained by re-indexing by re-defining $a := c$, $b := a-c$, and $c := a+b-c$. Next, we wish to show that the distinguished element $1 \in R_0$ is a unit with respect to this multiplication. Indeed, we have

$$1\cdot r \overset{(1)}{=} \sum_{a\in A} m_{0,a}(1, r_a) \overset{(2)}{=} \sum_{a\in A} r_a = r \qquad \text{and} \qquad r\cdot 1 \overset{(1)}{=} \sum_{a\in A} m_{a,0}(r_a, 1) \overset{(2)}{=} \sum_{a\in A} r_a = r,$$

where (1) follows by the fact that $m_{a,b}(0,-) = m_{a,b}(-,0) = 0$, which we have shown above, and (2) follows by unitality of the $m_{0,a}$'s and $m_{0,a}$'s, respectively. Finally, we wish to show that this product is distributive. Indeed, we have

$$\begin{aligned}
r\cdot(s+t) &= \sum_{a,b\in A} m_{a,b}(r_a, (s+t)_b)\\
&= \sum_{a,b\in A} m_{a,b}(r_a, s_b+t_b)\\
&\overset{(*)}{=} \sum_{a,b\in A} m_{a,b}(r_a, s_b) + \sum_{a,b\in A} m_{a,b}(r_a, t_b) = (r\cdot s) + (r\cdot t),
\end{aligned}$$

where $(*)$ follows by bilinearity of $m_{a,b}$. An entirely analagous argument yields that $(r+s)\cdot t = (r\cdot t) + (s\cdot t)$. □

Similarly, when defining A-graded modules, we will only define the action maps for homogeneous elements:

Lemma B.9. *Let R be an A-graded ring, M an A-graded abelian group, and suppose there exists $\mathbb{Z}$-bilinear maps $\kappa_{a,b} : R_a \times M_b \to M_{a+b}$ for all $a, b \in A$. Further suppose that for all $r \in R_a$, $r' \in R_b$, and $m \in M_c$ that*

$$\kappa_{a+b,c}(r\cdot r', m) = \kappa_{a,b+c}(r, \kappa_{b,c}(r', m)) \qquad \textit{and} \qquad \kappa_{0,c}(1, m) = m.$$

Then there is a unique map $\kappa : R\times M \to M$ which endows M with the structure of a left A-graded R-module and restricts to $\kappa_{a,b}$ for all $a, b \in A$.

On the other hand, suppose there exists $\mathbb{Z}$-bilinear maps $\kappa_{a,b} : M_a \times R_b \to M_{a+b}$ for all $a, b \in A$. Further suppose that for all $r \in R_a$, $r' \in R_b$, and $m \in M_c$ that

$$\kappa_{c,a+b}(m, r \cdot r') = \kappa_{c+a,b}(\kappa_{c,a}(m,r), r') \qquad \text{and} \qquad \kappa_{c,0}(m,1) = m.$$

Then there is a unique map $\kappa : M \times R \to M$ which endows M with the structure of a right A-graded R-module and restricts to $\kappa_{a,b}$ for all $a, b \in A$.

Finally, if we have maps $\lambda_{a,b} : R_a \times M_b \to M_{a+b}$ and $\rho_{a,b} : M_a \times R_b \to M_{a+b}$ satisfying all of the above conditions, and if we further have that

$$\lambda_{a,b+c}(r, \rho_{b,c}(x,s)) = \rho_{a+b,c}(\lambda_{a,b}(r,x), s)$$

for all $r \in R_a$, $x \in M_b$, and $s \in R_c$, then the left and right A-graded R-module structures induced on M by the λ's and ρ's give M the structure of an A-graded R-bimodule.

Proof. Checking this all is straightforward albeit tedious; we leave the proof as an exercise for the reader. $\square$

When working with A-graded rings and modules, we will often freely use the above propositions without comment.

Lemma B.10. *Let R be an A-graded ring, and let M be an A-graded left (resp. right) R-module. Then for all $d \in A$, the evaluation map*

$$\begin{aligned} \mathrm{ev}_1 : \mathrm{Hom}_R^d(R, M) &\to M_d \\ \varphi &\mapsto \varphi(1) \end{aligned}$$

is an isomorphism of abelian groups.

Proof. We consider the case that M is a left A-graded R-module, as showing it when M is a right module is entirely analagous. First of all, this map is clearly a homomorphism, as given degree d A-graded homomorphisms $\varphi, \psi : R \to M$, we have

$$\mathrm{ev}_1(\varphi + \psi) = (\varphi + \psi)(1) = \varphi(1) + \psi(1) = \mathrm{ev}_1(\varphi) + \mathrm{ev}_1(\psi).$$

Now, to see it is surjective, let $m \in M_d$, and define $\varphi_m : R \to M$ to send $r \mapsto r \cdot m$. First of all, φ_m is a module homomorphism, as given $r, s \in R$,

$$\varphi_m(r+s) = (r+s) \cdot m = r \cdot m + s \cdot m = \varphi_m(r) + \varphi_m(s) \quad \text{and} \quad \varphi_m(r \cdot s) = r \cdot s \cdot m = r \cdot \varphi_m(s).$$

Furthermore, it is clearly A-graded of degree d, as given a homogeneous element $r \in R_a$ for some $a \in A$, we have $\varphi_m(r) = r \cdot m \in R_{a+d}$, since m is homogeneous of degree d. Finally, clearly

$$\mathrm{ev}_1(\varphi_m) = \varphi_m(1) = 1 \cdot m = m,$$

so indeed ev_1 is surjective. On the other hand, to see it is injective, suppose we are given $\varphi, \psi \in \mathrm{Hom}_R^d(R, M)$ such that $\varphi(1) = \psi(1)$. Then given $r \in R$, we must have

$$\varphi(r) = \varphi(r \cdot 1) = r \cdot \varphi(1) = r \cdot \psi(1) = \psi(r \cdot 1) = \psi(r),$$

so φ and ψ are exactly the same map. Thus, ev_1 is injective, as desired. $\square$

B.2. **Tensor products of A-graded modules.**

Lemma B.11. *Given an A-graded ring R and two left (resp. right) A-graded R-modules M and N, their direct sum $M \oplus N$ is naturally a left (resp. right) A-graded R-module by defining*

$$(M \oplus N)_a := M_a \oplus N_a.$$

Proof. The canonical map $\bigoplus_{a \in A}(M_a \oplus N_a) \to M \oplus N$ factors as

$$\bigoplus_{a \in A}(M_a \oplus N_a) \xrightarrow{\cong} \bigoplus_{a \in A} M_a \oplus \bigoplus_{a \in A} N_a \xrightarrow{\cong} M \oplus N. \qquad \square$$

Recall that given a ring R, a left R-module M, a right R-module N, and an abelian group A, an *R-balanced map* $\varphi : M \times N \to B$ is one which satisifies

$$\begin{aligned}\varphi(m, n+n') &= \varphi(m,n) + \varphi(m,n')\\ \varphi(m+m', n) &= \varphi(m,n) + \varphi(m',n)\\ \varphi(m \cdot r, n) &= \varphi(m, r\cdot n)\end{aligned}$$

for all $m, m' \in M$, $n, n' \in N$, and $r \in R$. Then the tensor product $M \otimes_R N$ is the universal abelian group equipped with an R-balanced map $\otimes : M \times N \to M \otimes_R N$ such that for every abelian group B and every R-balanced map $\varphi : M \times N \to B$, there is a *unique* group homomorphism $\widetilde{\varphi} : M \otimes_R N \to B$ such that $\widetilde{f} \circ \otimes = f$. We call elements in the image of $\otimes : M \times N \to M \otimes_R N$ *pure tensors*. It is a standard fact that $M \otimes_R N$ is generated as an abelian group by its pure tensors.

Definition B.12. Suppose we have a right A-graded R-module M, a left A-graded R-module N, and an A-graded abelian group B. Then an *A-graded R-balanced map* $\varphi : M \times N \to B$ is an R-balanced map which restricts to $M_a \times N_b \to B_{a+b}$ for all $a, b \in A$.

Proposition B.13. *Suppose we have a right A-graded R-module M and a left A-graded R-module N. Then the tensor product*

$$M \otimes_R N$$

naturally inherits the structure of an A-graded abelian group by defining $(M \otimes_R N)_a$ to be the subgroup generated by homogeneous *pure tensors, i.e., those elements $m \otimes n$ with $m \in M_b$ and $n \in N_c$ such that $b + c = a$. Furthermore, if either M (resp. N) is an A-graded bimodule, then this decomposition makes $M \otimes_R N$ into a left (resp. right) A-graded R-module. In particular, if both M and N are R-bimodules, then $M \otimes_R N$ is an A-graded R-bimodule.*

Proof. By definition, since M and N are A-graded abelian groups, they are generated (as abelian groups) by their homogeneous elements. Thus it follows that $M \otimes_R N$ is generated by its homogeneous pure tensors, as defined above. Now, given a homogeneous pure tensor $m \otimes n$ in $M \otimes_R N$, it is clear that defining its degree by the formula $|m \otimes n| := |m| + |n|$ is perfectly well-defined, as given homogeneous elements $m \in M$, $n \in N$, and $r \in R$ we have that

$$|(m \cdot r) \otimes n| = |m \cdot r| + |n| = |m| + |r| + |n| = |m| + |r \cdot n| = |m \otimes (r \cdot n)|.$$

Thus, we may define $(M \otimes_R N)_a$ to be the subgroup of $M \otimes_R N$ generated by those pure homogeneous tensors of degree a. Now, consider the map

$$\Psi : M \times N \to \bigoplus_{a \in A} (M \otimes_R N)_a$$

which takes a pair $(m, n) = \sum_{a\in A}(m_a, n_a)$ to the element $\Psi(m, n)$ whose a^{th} component is

$$(\Psi(m,n))_a := \sum_{b+c=a} m_b \otimes n_c.$$

It is straightforward to see that this map is R-balanced, in the sense that it is additive in each argument and $\Psi(m \cdot r, n) = \Psi(m, r \cdot n)$ for all $m \in M$, $n \in N$, and $r \in R$. Thus by the universal property of $M \otimes_R N$, we get a homomorphism of abelian groups $\widetilde{\Psi} : M \otimes_R N \to \bigoplus_{a\in A}(M \otimes_R N)_a$ lifting Ψ along the canonical map $M \times N \to M \otimes_R N$. Now, also consider the canonical map

$$\Phi : \bigoplus_{a \in A} (M \otimes_R N)_a \to M \otimes_R N.$$

We would like to show $\widetilde{\Psi}$ and Φ are inverses of eah other. Since $\widetilde{\Psi}$ and Φ are both homomorphisms, it suffices to show this on generators. Let $m \otimes n$ be a homogeneous pure tensor with $m = m_a \in M_a$

and $n = n_b \in N_b$. Then we have

$$\Phi(\widetilde{\Psi}(m \otimes n)) = \Phi\left(\bigoplus_{a\in A}\sum_{b+c=a} m_b \otimes n_c\right) \overset{(*)}{=} \Phi(m \otimes n) = m \otimes n,$$

and

$$\widetilde{\Psi}(\Phi(m \otimes n)) = \widetilde{\Psi}(m \otimes n) = \bigoplus_{a\in A}\sum_{b+c=a} m_b \otimes n_c \overset{(*)}{=} m \otimes n,$$

where both occurrences of $(*)$ follow by the fact that $m_b \otimes n_c = 0$ unless $b = c = a$, in which case $m_a \otimes n_a = m \otimes n$. Thus since Φ is an isomorphism, $M \otimes_R N$ is indeed an A-graded abelian group, as desired.

Now, suppose that M is an A-graded R-bimodule, so there exists left and right A-graded actions of R on M such that given $r, s \in R$ and $m \in M$ we have $r \cdot (m \cdot s) = (r \cdot m) \cdot s$. Then we would like to show that given a left A-graded R-module N that $M \otimes_R N$ is canonically a left A-graded R-module. Indeed, define the action of R on $M \otimes_R N$ on pure tensors by the formula

$$r \cdot (m \otimes n) = (r \cdot m) \otimes n.$$

First of all, clearly this map is A-graded, as if $r \in R_a$, $m \in M_b$, and $n \in N_c$ then $(r \cdot m) \otimes n$, by definition, has degree $|r \cdot m| + |n| = |r| + |m| + |n|$ (the last equality follows since the left action of R on M is A-graded). In order to show the above map defines a left module structure, it suffices to show that given pure tensors $m \otimes n, m' \otimes n' \in M \otimes_R N$ and elements $r, r' \in R$ that

(1) $r \cdot (m \otimes n + m' \otimes n') = r \cdot (m \otimes n) + r \cdot (m' \otimes n')$,
(2) $(r + r') \cdot (m \otimes n) = r \cdot (m \otimes n) + r' \cdot (m' \otimes n')$,
(3) $(rr') \cdot (m \otimes n) = r \cdot (r' \cdot (m \otimes n))$, and
(4) $1 \cdot (m \otimes n) = m \otimes n$.

Axiom (1) holds by definition. To see (2), note that by the fact that R acts on M on the left that

$$(r + r') \cdot (m \otimes n) = ((r + r') \cdot m) \otimes n = (r \cdot m + r' \cdot m) \otimes n = r \cdot m \otimes n + r' \cdot m \otimes n.$$

That (3) and (4) hold follows similarly by the fact that $(rr') \cdot m = r \cdot (r' \cdot m)$ and $1 \cdot m = m$.

Conversely, if N is an A-graded R-bimodule, then showing $M \otimes_R N$ is canonically a right A-graded R-module via the rule

$$(m \otimes n) \cdot r = m \otimes (n \cdot r)$$

is entirely analagous.

Finally, if both M and N are R-bimodules, then by what we have shown, $M \otimes_R N$ is both a left and right R-module. To see these coincide to give $M \otimes_R N$ an R-bimodule structure, note that given $m \in M$, $n \in N$, and $r, r' \in R$ that

$$(r \cdot (m \otimes n)) \cdot r' = ((r \cdot m) \otimes n) \cdot r' = (r \cdot m) \otimes (n \cdot r') = r \cdot (m \otimes (n \cdot r')) = r \cdot ((m \otimes n) \cdot r'). \ \square$$

Lemma B.14. *Let R be an A-graded ring, B an A-graded abelian group, M a right A-graded R-module, and N a left A-graded R-module. Further suppose we are given a map $\varphi_{a,b} : M_a \times N_b \to B_{a+b}$ for all $a, b \in A$ which commutes with addition in each argument, and such that for all $m \in M_a$, $n \in N_b$, and $r \in R_c$ that*

$$\varphi_{a+b,c}(m \cdot r, n) = \varphi_{a,b+c}(m, r \cdot n).$$

Then there is a unique A-graded R-balanced map $\varphi : M \times N \to B$ which restricts to $\varphi_{a,b}$ for all $a, b \in A$, and furthermore, the induced homorphism $\widetilde{\varphi} : M \otimes_R N \to B$ is an A-graded homomorphism of abelian groups.

Proof. Checking this is straightforward, we leave it as an exercise for the reader. $\square$

B.3. **A-graded submodules and quotient modules.** In what follows, fix an A-graded ring R. We will simply say "A-graded R-module" when we are freely considering either left or right A-graded R-modules.

Recall that given a ring R, an R-module P is *projective* if, for all diagrams of R-module homomorphisms of the form

$$\begin{tikzcd} & M \arrow[d, two heads, "g"] \\ P \arrow[r, "f"'] & N \end{tikzcd}$$

with g an epimorphism, there exists a lift $h : P \to M$ satisfying $g \circ h = f$

$$\begin{tikzcd} & M \arrow[d, two heads, "g"] \\ P \arrow[r, "f"'] \arrow[ur, dashed, "h"] & N \end{tikzcd}$$

(Note h is not required to be unique.)

Definition B.15. Let R be an A-graded ring, and let P be an A-graded R-module. Then P is a *graded projective* module if, for all diagrams of A-graded R-module homomorphisms of the form

$$\begin{tikzcd} & M \arrow[d, two heads, "g"] \\ P \arrow[r, "f"'] & N \end{tikzcd}$$

with g an epimorphism, there exists an A-graded homomorphism $h : P \to M$ satisfying $g \circ h = f$.

$$\begin{tikzcd} & M \arrow[d, two heads, "g"] \\ P \arrow[r, "f"'] \arrow[ur, dashed, "h"] & N \end{tikzcd}$$

(Note h is not required to be unique.)

Definition B.16. Let M be an A-graded R-module. Then an *A-graded R-submodule* is an A-graded R-module N which is a subset of M and for which the inclusion $N \hookrightarrow M$ is an A-graded homomorphism of R-modules. Equivalently, it is a submodule N for which the canonical map

$$\bigoplus_{a \in A} N \cap M_a \to N$$

is an isomorphism.

Lemma B.17. *Let M be an A-graded R-module. Then an R-submodule $N \leq M$ is an A-graded submodule if and only if it is generated as an R-module by homogeneous elements of M.*

Proof. If $N \leq M$ is a A-graded submodule, it is generated by the set of all its homogeneous elements, which are also homogeneous elements in M, by definition.

Conversely, suppose $N \leq M$ is a submodule which is generated by homogeneous elements of M. Then define $N_a := N \cap M_a$, and consider the canonical map

$$\Phi : \bigoplus_{a \in A} N_a \to N.$$

First of all, it is surjective, as each generator of N belongs to some N_a, by definition. To see it is injective, consider the following commutative diagram:

$$\begin{tikzcd} \bigoplus_{a\in A} N_a \arrow[r, hook] \arrow[d, "\Phi"'] & \bigoplus_{a\in A} M_a \arrow[d, "\cong"] \\ N \arrow[r, hook] & M \end{tikzcd}$$

Since Φ composes with an injection to get an injection, clearly Φ must be injective itself. We have the desired result. ☐

Proposition B.18. *Given two left (resp. right) A-graded R-modules M and N and an A-graded R-module homomorphism $\varphi : M \to N$ (of possibly nonzero degree), the kernel and images of φ are A-graded submodules of M and N, respectively.*

Proof. First recall that a degree d A-graded homomorphism $M \to N$ is simply an A-graded homomorphism $M_* \to N_{*+d}$, so it suffices to consider the case φ is of degree 0. Next, note that since the forgetful functor from R-modules to abelian groups preserves kernels and images, it suffices to consider the case that φ is a homomorphism of A-graded abelian groups. Finally, by Lemma B.17, it suffices to show that $\ker\varphi$ and $\operatorname{im}\varphi$ are generated by homogeneous elements of M and N, respectively.

Note that by the universal property of the coproduct in **Ab**, the data of an A-graded homomorphism of abelian groups $\varphi : M \to N$ is precisely the data of an A-indexed collection of abelian group homomorphisms $\varphi_a : M_a \to N_a$, in which case the following diagram commutes:

$$\begin{tikzcd} \bigoplus_a M_a \arrow[r, "\bigoplus_a \varphi_a"] \arrow[d, "\cong"'] & \bigoplus_a N_a \arrow[d, "\cong"] \\ M \arrow[r, "\varphi"] & N \end{tikzcd}$$

Finally, the desired result follows by the purely formal fact that taking images and kernels commutes with arbitrary direct sums. ☐

Proposition B.19. *Given two left (resp. right) A-graded R-modules M and N, an A-graded submodule $K \leq N$, and an A-graded R-module homomorphism $\varphi : M \to N$ (of possibly nonzero degree), the submodule $\varphi^{-1}(K)$ of M is A-graded.*

Proof. Recall that a degree d A-graded homomorphism $M \to N$ is simply an A-graded homomorphism $M_* \to N_{*+d}$, so it suffices to consider the case φ is of degree 0. Now, let $x \in L := \varphi^{-1}(K)$. As an element of M, we may uniquely write $x = \sum_{a\in A} x_a$ where each $x_a \in M_a$. Similarly, if we set $y := \varphi(x)$, then we may uniquely write $y = \sum_{a\in A} y_a$ where each $y_a \in N_a$. Then since K is an A-graded submodule of N and $y \in K$, by definition, we have that $y_a \in K$ for each a. Finally, note that

$$\sum_{a\in A} y_a = y = \varphi(x) = \sum_{a\in A} \varphi(x_a),$$

so that $\varphi(x_a) = y_a \in K$ for all $a \in A$, so that $x_a \in L$ for all $a \in A$. Thus we have shown that each element in L can be written as a sum of homogeneous elements in M, as desired. ☐

Proposition B.20. *Given an A-graded R-module M and an A-graded subgroup $N \leq M$, the quotient M/N is canonically A-graded by defining $(M/N)_a$ to be the subgroup generated by cosets represented by homogeneous elements of degree a in M. Furthermore, the canonical maps $M_a/N_a \to (M/N)_a$ taking a coset $m + N_a$ to $m + N$ are isomorphisms.*

Proof. Consider the canonical map

$$\Phi : \bigoplus_a (M/N)_a \to M/N.$$

First of all, surjectivity of Φ follows by commutativity of the following diagram:

$$\begin{tikzcd} \bigoplus_a M_a \arrow[r, "\cong"] \arrow[d, two heads] & M \arrow[d, two heads] \\ \bigoplus_a (M/N)_a \arrow[r, "\Phi"] & M/N \end{tikzcd}$$

where the vertical left map sends a generator $m \in M_a$ to the coset $m + N$ in $(M/N)_a \subseteq M/N$. To see Φ is injective, suppose we are given some element $(m_a + N)_{a\in A}$ in $\bigoplus_a (M/C)_a$ such that $\sum_{a\in A}(m_a + N) = 0$ in M/N. Thus $\sum_{a\in A} m_a \in N$, and since N is A-graded this implies that each m_a belongs to $N \cap M_a = N_a$, so that in particular $m_a + N$ is zero in $(M/N)_a \subseteq M/N$, so that $(m_a + N)_{a\in A} = 0$ in $\bigoplus_a (M/N)_a$, as desired.

It remains to show that the canonical map

$$\varphi_a : M_a/N_a \to (M/N)_a$$

is an isomorphism. It is clearly surjective, as $(M/N)_a$ is generated by elements $m + N$ for $m \in M_a$, and these elements make up precisely the image of φ_a. Thus φ_a hits every generator of $(M/N)_a$, so φ_a is surjective. On the other hand, suppose we are given some $m \in M_a$ such that $\varphi(m + N_a) = m + N = 0$. Thus $m \in N$, and $m \in M_a$, so that $m \in M_a \cap N = N_a$, meaning $m + N_a = 0$ in M_a/N_a, as desired. □

B.4. **Pushouts of A-graded anticommutative rings.** The goal of this section is to show that given an A-graded anticommutative ring R (Definition 4.5) that the category R-$\mathbf{GrCAlg}(A)$ of A-graded anticommutative R-algebras (Definition 4.6) has pushouts and binary coproducts, which are formed by taking the tensor product of the underlying A-graded modules and endowing it with an anticommutative product. The proofs here are entirely analagous to showing that the standard category of anticommutative $\mathbb{Z}$-graded rings has pushouts, so rather than giving complete proofs in this section we simply outline what needs to be shown, and leave it to the reader to fill in the details.

Proposition B.21. *Suppose we have an A-graded anticommutative ring R (Definition 4.5) and two morphisms $f : (B, \varphi_B) \to (C, \varphi_C)$ and $g : (B, \varphi_B) \to (D, \varphi_D)$ in R-$\mathbf{GrCAlg}(A)$ (Definition 4.6). Then f and g make C and D both B-bimodules, respectively,*[7] *so we may form their tensor product $C \otimes_B D$, which is itself an A-graded B-bimodule (Proposition B.13). Then $C \otimes_B D$ canonically inherits the structure of an A-graded R-commutative ring with unit $1_C \otimes 1_D$ via a product*

$$(C \otimes_B D) \times (C \otimes_B D) \to C \otimes_B D$$

which sends a pair $(x \otimes y, x' \otimes y')$ of homogeneous pure tensors to the element

$$\varphi_B(\theta_{|x|,|y'|}) \cdot (xx' \otimes yy') = \varphi_C(\theta_{|x|,|y'|})xx' \otimes yy',$$

(where here $\cdot$ denotes the left module action of B on $C \otimes_B D$), and with structure map

$$\begin{aligned} \varphi : R &\to C \otimes_B D \\ r &\mapsto \varphi_B(r) \cdot (1_C \otimes 1_D) = (\varphi_C(r) \otimes 1_D) = (1_C \otimes \varphi_D(r)). \end{aligned}$$

[7]Explicitly, it is a standard fact that given a ring homomorphism $\varphi : R \to S$ that S canonically becomes an R-bimodule with left action $r \cdot s := \varphi(r)s$ and right action $s \cdot r := s\varphi(r)$, so that in particular if φ is an A-graded homomorphism of A-graded rings, then φ makes S an A-graded R-bimodule.

Proof sketch. We simply lay out everything that needs to be shown, and we leave it to the reader to fill in the details. First to show that the indicated product is actually well-defined and distributive, by Lemma B.14 it suffices to show that for all homogeneous $c, c', c'' \in C$, $d, d', d'' \in D$, and $b \in B$ with $|c'| = |c''|$ and $|d'| = |d''|$, that

$$\begin{aligned}
\varphi_B(\theta_{|d|,|c'+c''|}) \cdot (c(c'+c'') \otimes dd') &= \varphi_B(\theta_{|d|,|c'|}) \cdot (cc' \otimes dd') + \varphi_B(\theta_{|d|,|c''|}) \cdot (cc'' \otimes dd') \\
\varphi_B(\theta_{|d|,|c'|}) \cdot (cc' \otimes d(d'+d'')) &= \varphi_B(\theta_{|d|,|c'|}) \cdot (cc' \otimes dd') + \varphi_B(\theta_{|d|,|c'|}) \cdot (cc' \otimes dd'') \\
\varphi_B(\theta_{|d|,|c'\cdot b|}) \cdot (c(c' \cdot b) \otimes dd') &= \varphi_B(\theta_{|d|,|c'|}) \cdot (cc' \otimes d(b \cdot d')) \\
\varphi_B(\theta_{|d'|,|c|}) \cdot ((c'+c'')c \otimes d'd) &= \varphi_B(\theta_{|d'|,|c|}) \cdot (c'c \otimes d'd) + \varphi_B(\theta_{|d'|,|c|}) \cdot (c''c \otimes d'd) \\
\varphi_B(\theta_{|d'+d''|,|c|}) \cdot (c'c \otimes (d'+d'')d) &= \varphi_B(\theta_{|d'|,|c|}) \cdot (c'c \otimes d'd) + \varphi_B(\theta_{|d''|,|c|}) \cdot (c'c \otimes d''d) \\
\varphi_B(\theta_{|d'|,|c|})((c' \cdot b)c \otimes d'd) &= \varphi_B(\theta_{|c|,|b\cdot d'|}) \cdot (c'c \otimes (b \cdot d')d),
\end{aligned}$$

where each occurrence of $\cdot$ denotes the left or right module action of B. These tell us that for all $x \in C \otimes_B D$ that the maps $C \otimes_B D \to C \otimes_B D$ sending $y \mapsto xy$ and $y \mapsto yx$ are well-defined A-graded homomorphisms of abelian groups, so we have a distributive product $(x, y) \mapsto xy$. Then to show that this product makes $C \otimes_B D$ an A-graded ring, we need to show it is associative and unital. By Lemma B.8, it suffices to show that for all *homogeneous* $x, y, z \in C \otimes_B D$ that $(xy)z = x(yz)$ and $x(1_C \otimes 1_D) = x = (1_C \otimes 1_D)x$. By distributivity, it further suffices to consider the case that x, y, and z are homogeneous *pure tensors* in $C \otimes_B D$, i.e., it suffices to show that for all homogeneous $c, c', c'' \in C$ and $d, d', d'' \in D$ that

$$((c \otimes d)(c' \otimes d'))(c'' \otimes d'') = (c \otimes d)((c' \otimes d')(c'' \otimes d''))$$

and

$$(c \otimes d)(1_C \otimes 1_D) = (c \otimes d) = (1_C \otimes 1_D)(c \otimes d).$$

Thus, proving these hold will show $C \otimes_B D$ has the structure of an A-graded ring, as desired. Now, we wish to show that the given map $\varphi : R \to C \otimes_B D$ is a ring homomorphism. Clearly it sends 1 to $1_C \otimes 1_D$, and again by linearity, it suffices to show that given *homogeneous* $r, s \in R$ that

$$\varphi(r+s) = \varphi_B(r+s)(1_C \otimes 1_D) = \varphi_B(r)(1_C \otimes 1_D) + \varphi_B(s)(1_C \otimes 1_D) = \varphi(r) + \varphi(s)$$

and

$$\varphi(rs) = \varphi_B(rs)(1_C \otimes 1_D) = (\varphi_B(r)(1_C \otimes 1_D))(\varphi_B(s)(1_C \otimes 1_D)) = \varphi(r)\varphi(s).$$

Finally, we need to show that $C \otimes_B D$ satisfies the graded commutativity condition, for which again by linearity it suffices to show that given homogeneous $c, c' \in C$ and $d, d' \in D$ that

$$(c \otimes d)(c' \otimes d') = \varphi(\theta_{|c\otimes d|,|c'\otimes d'|})(c' \otimes d')(c \otimes d) = \varphi(\theta_{|c|+|d|,|c'|+|d'|})(c' \otimes d')(c \otimes d).$$

Showing all of these is relatively straightforward. ☐

Proposition B.22. *Given an A-graded anticommutative ring (R, θ), the category R-$\mathbf{GrCAlg}(A)$ has pushouts, where given $f : (B, \varphi_B) \to (C, \varphi_C)$ and $g : (B, \varphi_B) \to (D, \varphi_D)$, their pushout is the object $(C \otimes_B D, \varphi)$ constructed in Proposition B.21, along with the canonical maps $(C, \varphi_C) \to (C \otimes_B D, \varphi)$ sending $c \mapsto c \otimes 1_D$ and $(D, \varphi_D) \to (C \otimes_B D, \varphi)$ sending $d \mapsto 1_C \otimes d$. In particular, since (R, id_R) is initial, R-$\mathbf{GrCAlg}(A)$ has binary coproducts.*

Proof sketch. First, we need to show that the given maps $i_C : (C, \varphi_C) \to (C \otimes_B D, \varphi)$ and $i_D : (D, \varphi_D) \to (C \otimes_B D, \varphi)$ are actually morphisms in R-$\mathbf{GrCAlg}(A)$, i.e., that they are ring homomorphisms and that the following diagram commutes:

$$\begin{tikzcd}
 & R \arrow[dl, "\varphi_C"'] \arrow[d, dashed, "\varphi"] \arrow[dr, "\varphi_D"] & \\
C \arrow[r, "i_C"'] & C \otimes_B D & D \arrow[l, "i_D"]
\end{tikzcd}$$

Showing this is entirely straightforward. Furthermore, i_C and i_D clearly make the following diagram commute:

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "f"'] & D \arrow[d, "i_D"] \\ C \arrow[r, "i_C"'] & C \otimes_B D \end{tikzcd}$$

It remains to show that i_C and i_D are the universal such arrows. Suppose we have some object (E, φ_E) in R-**GrCAlg**(A) and a commuting diagram

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "f"'] & D \arrow[d, "k"] \\ C \arrow[r, "h"] & E \end{tikzcd}$$

of morphisms in R-**GrCAlg**(A). Then we'd like to show there exists a unique morphism $\ell : C \otimes_B D \to E$ in R-**GrCAlg**(A) which makes the following diagram commute:

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "f"'] & D \arrow[d, "i_D"'] \arrow[ddr, bend left, "k"] & \\ C \arrow[r, "i_C"] \arrow[drr, bend right, "h"'] & C \otimes_B D \arrow[dr, dashed, "\ell"] & \\ & & E \end{tikzcd}$$

First we show uniqueness. Supposing such an arrow ℓ existed, given elements $c \in C$ and $d \in D$, we must have

$$\ell(c \otimes d) = \ell((c \otimes 1_D)(1_C \otimes d)) = \ell(c \otimes 1_D)\ell(1_C \otimes d) = \ell(i_C(c))\ell(i_D(d)) = h(c)k(d).$$

Since pure tensors generate $C \otimes_B D$, we have uniquely determined ℓ, and clearly it makes the above diagram commute. Now, it remains to show that as defined ℓ is a morphism in R-**GrCAlg**(A), i.e., that it is an A-graded ring homomorphism and that the following diagram commutes:

$$\begin{tikzcd} & R \arrow[dl, "\varphi"'] \arrow[dr, "\varphi_E"] & \\ C \otimes_B D \arrow[rr, "\ell"] & & E \end{tikzcd}$$

This is all entirely straightforward to show. $\square$

Appendix C. Monoid objects

In this appendix, we fix a symmetric monoidal category $(\mathcal{C}, \otimes, S)$ with left unitor, right unitor, associator, and symmetry isomorphisms λ, ρ, α, and τ, respectively.

C.1. **Monoid objects in a symmetric monoidal category.**

Definition C.1. A *monoid object* (E, μ, e) is an object E in $\mathcal{C}$ along with a multiplication morphism $\mu : E \otimes E \to E$ and a unit map $e : S \to E$ such that the following diagrams commute:

$$\begin{tikzcd} E \otimes S \arrow[r, "E\otimes e"] \arrow[dr, "\rho_E"'] & E \otimes E \arrow[d, "\mu"] & S \otimes E \arrow[l, "e\otimes E"'] \arrow[dl, "\lambda_E"] \\ & E & \end{tikzcd} \qquad \begin{tikzcd} (E \otimes E) \otimes E \arrow[rr, "\mu\otimes E"] \arrow[d, "\alpha"'] & & E \otimes E \arrow[d, "\mu"] \\ E \otimes (E \otimes E) \arrow[r, "E\otimes\mu"] & E \otimes E \arrow[r, "\mu"] & E \end{tikzcd}$$

The first diagram expresses unitality, while the second expressed associativity. If in addition the following diagram commutes,

$$\begin{tikzcd} E\otimes E \arrow[rr, "\tau"] \arrow[dr, "\mu"'] & & E\otimes E \arrow[dl, "\mu"] \\ & E & \end{tikzcd}$$

then we say (E,μ,e) is a *commutative* monoid object.

Example C.2. The object S is a monoid object, with multiplication map $\rho_S=\lambda_S : S\otimes S\to S$ and unit $\mathrm{id}_S : S\to S$.

Definition C.3. Given two monoid objects (E_1,μ_1,e_1) and (E_2,μ_2,e_2) in a symmetric monoidal category $(\mathcal{C},\otimes,S)$, a *monoid homomorphism* from E_1 to E_2 is a morphism $f : E_1\to E_2$ in $\mathcal{C}$ such that the following diagrams commute:

$$\begin{tikzcd} E_1\otimes E_1 \arrow[r, "f\otimes f"] \arrow[d, "\mu_1"'] & E_2\otimes E_2 \arrow[d, "\mu_2"] \\ E_1 \arrow[r, "f"] & E_2 \end{tikzcd} \qquad \begin{tikzcd} & S \arrow[dl, "e_1"'] \arrow[dr, "e_2"] & \\ E_1 \arrow[rr, "f"] & & E_2 \end{tikzcd}$$

It is straightforward to show that id_{E_1} is a homomorphism of monoid objects from E_1 to itself, and that the composition of monoid homomorphisms is still a monoid homomorphism. Thus, we have categories $\mathbf{Mon}_{\mathcal{C}}$ and $\mathbf{CMon}_{\mathcal{C}}$ of monoid objects and commutative monoid objects in $\mathcal{C}$, respectively, with monoid homomorphisms between them.

Lemma C.4. *Given two monoid objects (E_1,μ_1,e_1) and (E_2,μ_2,e_2) in a symmetric monoidal category $(\mathcal{C},\otimes,S)$, their tensor product $E_1\otimes E_2$ canonically becomes a monoid object in $\mathcal{C}$ with unit map*

$$e : S \xrightarrow{\cong} S\otimes S \xrightarrow{e_1\otimes e_2} E_1\otimes E_2$$

and multiplication map

$$\mu : E_1\otimes E_2\otimes E_1\otimes E_2 \xrightarrow{E_1\otimes\tau\otimes E_2} E_1\otimes E_1\otimes E_2\otimes E_2 \xrightarrow{\mu_1\otimes\mu_2} E_1\otimes E_2$$

(where here we are suppressing the associators from the notation). If in addition (E_1,μ_1,e_1) and (E_2,μ_2,e_2) are commutative *monoid objects, then $(E_1\otimes E_2,\mu,e)$ is as well.*

Proof. Due to the size of the diagrams involved, we leave this as an exercise for the reader. It is entirely straightforward. □

Lemma C.5. *Given monoid objects (E_i,μ_i,e_i) for $i=1,2,3$ in a symmetric monoidal category $\mathcal{C}$, the associator $(E_1\otimes E_2)\otimes E_3 \xrightarrow{\cong} E_1\otimes(E_2\otimes E_3)$ is an isomorphism of monoid objects. In other words, up to associativity, given a collection of monoid objects $E_1,\dots,E_n$ in $\mathcal{C}$, there is no ambiguity when talking about their tensor product $E_1\otimes\cdots\otimes E_n$ as a monoid object.*

Proof. Clearly, up to associativity, $(E_1\otimes E_2)\otimes E_3$ and $E_1\otimes(E_2\otimes E_3)$ have the same unit map $S\xrightarrow{e_1\otimes e_2\otimes e_3} E_1\otimes E_2\otimes E_3$. Thus, it remains to show that they have the same product map, up to associativity. To see this, consider the following diagram, where we've passed to a symmetric strict monoidal category:

$$\begin{tikzcd}
E_1 \otimes (E_2 \otimes E_3) \otimes E_1 \otimes (E_2 \otimes E_3) \arrow[rrrr, equal, "\alpha"] \arrow[d, "E_1 \otimes \tau_{E_2 \otimes E_3, E_1} \otimes E_2 \otimes E_3"'] & & & & (E_1 \otimes E_2) \otimes E_3 \otimes (E_1 \otimes E_2) \otimes E_3 \arrow[d, "E_1 \otimes E_2 \otimes \tau_{E_3, E_1 \otimes E_2} \otimes E_3"] \\
E_1 \otimes E_1 \otimes E_2 \otimes E_3 \otimes E_2 \otimes E_3 \arrow[d, "\mu_1 \otimes E_2 \otimes \tau \otimes E_3"'] \arrow[drr, "E_1 \otimes E_1 \otimes E_2 \otimes \tau \otimes E_3"] & & & & E_1 \otimes E_2 \otimes E_1 \otimes E_2 \otimes E_3 \otimes E_3 \arrow[d, "E_1 \otimes \tau \otimes E_2 \otimes \mu_3"] \arrow[dll, "E_1 \otimes \tau \otimes E_2 \otimes E_3 \otimes E_3"'] \\
E_1 \otimes E_2 \otimes E_2 \otimes E_3 \otimes E_3 \arrow[d, "E_1 \otimes \mu_2 \otimes \mu_3"'] & & E_1 \otimes E_1 \otimes E_2 \otimes E_2 \otimes E_3 \otimes E_3 \arrow[ll, "\mu_1 \otimes E_2 \otimes E_2 \otimes E_3 \otimes E_3"] \arrow[rr, "E_1 \otimes E_2 \otimes E_2 \otimes E_2 \otimes \mu_3"'] \arrow[dll, "\mu_1 \otimes \mu_2 \otimes \mu_3"] \arrow[drr, "\mu_1 \otimes \mu_2 \otimes \mu_3"'] & & E_1 \otimes E_1 \otimes E_2 \otimes E_2 \otimes E_3 \arrow[d, "\mu_1 \otimes \mu_2 \otimes E_3"] \\
E_1 \otimes E_2 \otimes E_3 \arrow[rrrr, equal, "\alpha"] & & & & E_1 \otimes E_2 \otimes E_3
\end{tikzcd}$$

The top pentagonal region commutes by coherence for the τ's in a symmetric monoidal category. The bottom triangle commutes by definition. The remaining four triangles commute by functoriality of $- \otimes -$. On the left is the product for $E_1 \otimes (E_2 \otimes E_3)$, while on the right is the product for $(E_1 \otimes E_2) \otimes E_3$. Thus they are equal up to associativity, as desired. $\square$

Lemma C.6. *Let (E, μ, e) be a monoid object in $\mathcal{SH}$. Then the map $e : S \to E$ is a monoid homomorphism. Furthermore, if E is a* commutative *monoid object, then $\mu : E \otimes E \to E$ is also a monoid object homomorphism. (Here S and $E \otimes E$ are considered to be monoid objects by Example C.2 and Lemma C.4, respectively.)*

Proof. To see e is a monoid homomorphism, consider the following diagrams:

$$\begin{tikzcd}
& S \arrow[dl, equal] \arrow[dr, "e"] & \\
S \arrow[rr, "e"] & & E
\end{tikzcd}
\qquad
\begin{tikzcd}
S \otimes S \arrow[rr, "e \otimes e"] \arrow[dr, dashed, "S \otimes e"] \arrow[dd, "\rho_S = \lambda_S"'] & & E \otimes E \arrow[dd, "\mu"] \\
& S \otimes E \arrow[ur, dashed, "e \otimes E"'] \arrow[dr, "\lambda_E"] & \\
S \arrow[rr, "e"'] & & E
\end{tikzcd}$$

The left diagram commutes by definition. The top region in the right diagram commutes by functoriality of $- \otimes -$. The right region commutes by unitality of μ. The left region commutes by naturality of λ. Thus, indeed $e : S \to E$ is a monoid object homomorphism.

Now, to see μ is a monoid object homomorphism when (E, μ, e) is a commutative monoid object, first consider the following diagram:

$$\begin{tikzcd}
& S \arrow[dl, "e \otimes e"'] \arrow[d, dashed, "e"] \arrow[dr, "e"] & \\
& E \arrow[dl, "E \otimes e"] \arrow[dr, equal] & \\
E \otimes E \arrow[rr, "\mu"'] & & E
\end{tikzcd}$$

The left region commutes by functoriality of $-\otimes-$, the right region commutes by definition, and the bottom region commutes by unitality of μ. Now, consider the following diagram:

$$\begin{tikzcd}
E_1\otimes E_2\otimes E_3\otimes E_4 \arrow[rrr, "\mu\otimes\mu"] \arrow[dr, "E\otimes\mu\otimes E"'] \arrow[drr, "\mu\otimes E\otimes E"] \arrow[dd, "E\otimes\tau\otimes E"'] & & & E_{12}\otimes E_{34} \arrow[ddd, "\mu"] \\
& E_1\otimes E_{23}\otimes E_4 \arrow[dr, "\mu\otimes E"] & E_{12}\otimes E_3\otimes E_4 \arrow[ur, "E\otimes\mu"] \arrow[d, "\mu\otimes E"] & \\
E_1\otimes E_3\otimes E_2\otimes E_4 \arrow[ur, "E\otimes\mu\otimes E"] \arrow[r, "\mu\otimes E\otimes E"] \arrow[d, "\mu\otimes\mu"'] & E_{13}\otimes E_2\otimes E_4 \arrow[r, "\mu\otimes E"'] \arrow[dl, "E\otimes\mu"] & E_{123}\otimes E_4 \arrow[dr, "\mu"] & \\
E_{13}\otimes E_{24} \arrow[rrr, "\mu"] & & & E_{1234}
\end{tikzcd}$$

Here we have numbered the E's to make it clearer what's going on. The top and bottom left regions commute by functoriality of $-\otimes-$. The top left region commutes by commutativity of μ. Every other region commutes by associativity of μ. Thus, we've shown μ is a monoid object homomorphism, as desired. $\square$

Lemma C.7. *Suppose we have some monoid object (E,μ,e) in $\mathcal{C}$ and some homomorphism of monoid objects $f:(E_1,\mu_1,e_1)\to(E_2,\mu_2,e_2)$ in $\mathbf{Mon}_{\mathcal{C}}$. Then $E\otimes f: E\otimes E_1\to E\otimes E_2$ and $f\otimes E: E_1\otimes E\to E_2\otimes E$ are monoid homomorphisms, where here we are considering $E\otimes E_1$, $E\otimes E_2$, $E_1\otimes E$, and $E_2\otimes E$ to be monoid objects by Lemma C.4.*

Proof. We will show that $E\otimes f$ is a monoid object homomorphism, as showing $f\otimes E$ is a monoid homomorphism is entirely analagous. First consider the following diagram:

$$\begin{tikzcd}
E\otimes E_1\otimes E\otimes E_1 \arrow[rrr, "E\otimes f\otimes E\otimes f"] \arrow[d, "E\otimes\tau\otimes E_1"'] & & & E\otimes E_2\otimes E\otimes E_2 \arrow[d, "E\otimes\tau\otimes E_2"] \\
E\otimes E\otimes E_1\otimes E_1 \arrow[rrr, "E\otimes E\otimes f\otimes f"] \arrow[dr, "\mu\otimes E_1\otimes E_2"] \arrow[dd, "\mu\otimes\mu_1"'] & & & E\otimes E\otimes E_2\otimes E_2 \arrow[dl, "\mu\otimes E_2\otimes E_2"'] \arrow[dd, "\mu\otimes\mu_2"] \\
& E\otimes E_1\otimes E_1 \arrow[r, "E\otimes f\otimes f"] \arrow[dl, "E\otimes\mu_1"] & E\otimes E_2\otimes E_2 \arrow[dr, "E\otimes\mu_2"] & \\
E\otimes E_1 \arrow[rrr, "E\otimes f"] & & & E\otimes E_2
\end{tikzcd}$$

The top region commutes by naturality of τ. The bottom trapezoid commutes since f is a monoid homomorphism. The remaining three regions commute by functoriality of $-\otimes-$. Now, consider the following diagram:

$$\begin{tikzcd}
& S \arrow[d, "e"] \arrow[ddl, "e\otimes e_1"'] \arrow[ddr, "e\otimes e_2"] & \\
& E \arrow[dl, "E\otimes e_1"] \arrow[dr, "E\otimes e_2"] & \\
E\otimes E_1 \arrow[rr, "E\otimes f"] & & E\otimes E_2
\end{tikzcd}$$

The bottom region commutes since f is a monoid homomorphism. The top two regions commute by functoriality of $-\otimes-$. Thus, we've shown $E\otimes f$ is a monoid object homomorphism, as desired. $\square$

C.2. **Modules over monoid objects in a symmetric monoidal category.**

Definition C.8. Let (E, μ, e) be a monoid object in $\mathcal{C}$. Then a *(left) module object* (N, κ) over (E, μ, e) is the data of an object N in $\mathcal{C}$ and a morphism $\kappa : E \otimes N \to N$ such that the following two diagrams commute in $\mathcal{C}$:

$$\begin{tikzcd} S \otimes N \arrow[r, "e\otimes N"] \arrow[dr, "\lambda_N"'] & E \otimes N \arrow[d, "\kappa"] \\ & N \end{tikzcd} \qquad \begin{tikzcd} (E \otimes E) \otimes N \arrow[rr, "\mu\otimes N"] \arrow[d, "\alpha"'] & & E \otimes N \arrow[d, "\kappa"] \\ E \otimes (E \otimes N) \arrow[r, "E\otimes\kappa"] & E \otimes N \arrow[r, "\kappa"] & N \end{tikzcd}$$

Definition C.9. Let (E, μ, e) be a monoid object in $\mathcal{C}$, and suppose we have two (left) module objects (N, κ) and (N', κ') over (E, μ, e). Then a morphism $f : N \to N'$ is a *(left) E-module homomorphism* if the following diagram commutes in $\mathcal{C}$:

$$\begin{tikzcd} E \otimes N \arrow[r, "E\otimes f"] \arrow[d, "\kappa"'] & E \otimes N' \arrow[d, "\kappa'"] \\ N \arrow[r, "f"] & N' \end{tikzcd}$$

Definition C.10. Given a monoid object (E, μ, e) in $\mathcal{C}$, we write E-**Mod** to denote the category of (left) module objects over E and E-module homomorphisms between them. We denote the homset in E-**Mod** by

$$\mathrm{Hom}_{E\text{-}\mathbf{Mod}}(M, N), \qquad \text{or simply} \qquad \mathrm{Hom}_E(M, N).$$

For our purposes, we will only consider left module objects, so we will usually drop the quanitfier "left" and just refer to them as "module objects".

Lemma C.11. *Let (E, μ, e) be a monoid object in $\mathcal{C}$ and let (N, κ) be an E module object. Then given some object X in $\mathcal{C}$ and an isomorphism $\phi : N \xrightarrow{\cong} X$, X inherits the structure of an E-module via the action map*

$$\kappa_\phi : E \otimes X \xrightarrow{E\otimes\phi^{-1}} E \otimes N \xrightarrow{\kappa} N \xrightarrow{\phi} X.$$

Proof. We need to show the two coherence diagrams in Definition C.8 commute. To see the former commutes, consider the following diagram:

$$\begin{tikzcd} X \arrow[rr, "e\otimes X"] \arrow[dr, "\phi^{-1}"] \arrow[dddrr, equal] & & E \otimes X \arrow[d, "E\otimes\phi^{-1}"] \\ & N \arrow[r, "e\otimes N"] \arrow[dr, equal] & E \otimes N \arrow[d, "\kappa"] \\ & & N \arrow[d, "\phi"] \\ & & X \end{tikzcd}$$

The top trapezoid commutes by functoriality of $- \otimes -$. The middle small triangle commutes by unitality of κ. The remaining region commutes by definition. To see the second coherence diagram commutes, consider the following diagram:

$$\begin{tikzcd}
E \otimes E \otimes X \arrow[rrr, "\mu\otimes X"] \arrow[d, "E\otimes E\otimes\phi^{-1}"'] & & & E \otimes X \arrow[d, "E\otimes\phi^{-1}"] \\
E \otimes E \otimes N \arrow[rrr, "\mu\otimes N"] \arrow[d, "E\otimes\kappa"'] & & & E \otimes N \arrow[d, "\kappa"] \\
E \otimes N \arrow[rrr, "\kappa"] \arrow[d, "E\otimes\phi"'] \arrow[dr, equal] & & & N \arrow[d, "\phi"] \\
E \otimes X \arrow[r, "E\otimes\phi^{1-}"'] & E \otimes N \arrow[r, "\kappa"'] & N \arrow[r, "\phi"'] & X
\end{tikzcd}$$

The top rectangle commutes by functoriality of $- \otimes -$. The middle rectangle commutes by coherence for κ. The bottom two regions commute by definition. $\square$

Proposition C.12. *Given a monoid object (E, μ, e) in $\mathcal{C}$, the forgetful functor E-**Mod** $\to \mathcal{C}$ has a left adjoint $\mathcal{C} \to E$-**Mod** sending an object X in $\mathcal{C}$ to $(E \otimes X, \kappa_X)$ where κ_X is the composition*

$$E \otimes (E \otimes X) \xrightarrow{\alpha^{-1}} (E \otimes E) \otimes X \xrightarrow{\mu\otimes X} E \otimes X,$$

and sending a morphism $f : X \to Y$ to $E \otimes f : E \otimes X \to E \otimes Y$.

*We call this functor $E \otimes - : \mathcal{C} \to E$-**Mod** the* free *functor, and we call E-modules in the image of the free functor* free modules*.*

Proof. In this proof, we work in a symmetric strict monoidal category. First, we wish to show that $E \otimes - : \mathcal{C} \to E$-**Mod** as constructed is well-defined. First, to see that (X, κ_X) is actually a E-module, we need to show the two diagrams in Definition C.8 commute. Indeed, consider the following diagrams:

$$\begin{tikzcd}
E \otimes X \arrow[r, "e\otimes E\otimes X"] \arrow[dr, equal] & E \otimes E \otimes X \arrow[d, "\mu\otimes X"] \\
& E \otimes X
\end{tikzcd}
\qquad
\begin{tikzcd}
E \otimes E \otimes E \otimes X \arrow[r, "\mu\otimes E\otimes X"] \arrow[d, "E\otimes\mu\otimes X"'] & E \otimes E \otimes X \arrow[d, "\mu\otimes X"] \\
E \otimes E \otimes X \arrow[r, "\mu\otimes X"'] & E \otimes X
\end{tikzcd}$$

These are precisely the diagrams obtained by applying $X \otimes -$ to the coherence diagrams for μ, so that they commute as desired. Now, suppose $f : X \to Y$ is a morphism in $\mathcal{C}$, then we would like to show that $E \otimes f : E \otimes X \to E \otimes Y$ is a morphism of E-module objects. Indeed, consider the following diagram:

$$\begin{tikzcd}
E \otimes E \otimes X \arrow[r, "E\otimes E\otimes f"] \arrow[d, "\mu\otimes X"'] & E \otimes E \otimes Y \arrow[d, "\mu\otimes Y"] \\
E \otimes X \arrow[r, "E\otimes f"] & E \otimes Y
\end{tikzcd}$$

It commutes by functoriality of $- \otimes -$, so $E \otimes f$ is indeed an E-module homomorphism as desired.

Now, in order to see that $E \otimes -$ is left adjoint to the forgetful functor, it suffices to construct a unit and counit for the adjunction and show they satisfy the zig-zag identities. Given X in $\mathcal{C}$ and (N, κ) in E-**Mod**, define $\eta_X := e \otimes X : X \to E \otimes X$ and $\varepsilon_{(N,\kappa)} := \kappa : E \otimes N \to N$. η_X is clearly natural in X by functoriality of $- \otimes -$, and $\varepsilon_{(N,\kappa)}$ is natural in (N, κ) by how morphisms in E-**Mod** are defined. Now, to see these are actually the unit and counit of an adjunction, we need to show that the following diagrams commute for all X in $\mathcal{C}$ and (N,κ) in E-**Mod**:

$$\begin{tikzcd}
E\otimes X \arrow[r, "E\otimes\eta_X = E\otimes e\otimes X"] \arrow[dr, equal] & E\otimes E\otimes X \arrow[d, "\varepsilon_{(E\otimes X,\kappa_X)}=\mu\otimes X"] \\
& E\otimes X
\end{tikzcd}
\qquad
\begin{tikzcd}
E\otimes N \arrow[d, "\varepsilon_{(N,\kappa)}=\kappa"'] & N \arrow[l, "\eta_N = e\otimes N"'] \arrow[dl, equal] \\
N &
\end{tikzcd}$$

Commutativity of the left diagram is unitality of μ, while commutativity of the right diagram is unitality of κ. Thus indeed $E\otimes - : \mathcal{C}\to E$-**Mod** is a left adjoint of the forgetful functor E-**Mod** $\to\mathcal{C}$, as desired. □

Lemma C.13. *Let (E,μ,e) be a monoid object in $\mathcal{C}$. Further suppose we have some object X in $\mathcal{C}$ and an E-module object (N,κ), along with a commuting diagram in $\mathcal{C}$*

$$\begin{tikzcd}
X \arrow[r, "\iota"'] \arrow[rr, bend left, equal] & N \arrow[r, "r"'] & X
\end{tikzcd}$$

Then if $\ell := \iota\circ r : N\to N$ is an E-module homomorphism, then X is canonically an E-module object with structure map

$$\kappa_X : E\otimes X \xrightarrow{E\otimes\iota} E\otimes N \xrightarrow{\kappa} N \xrightarrow{r} X,$$

and furthermore, the maps $\iota : X\to N$ and $r : N\to X$ are E-module homomorphisms.

Proof. First, in order to show (X,κ_X) is an E-module, we need to show the two diagrams in Definition C.8 commute. To see the unitality diagram holds, consider the following diagram:

$$\begin{tikzcd}
S\otimes X \arrow[rr, "e\otimes X"] \arrow[dr, "S\otimes\iota"] \arrow[ddd, "\lambda_X"'] & & E\otimes X \arrow[d, "E\otimes\iota"] \\
& S\otimes N \arrow[r, "e\otimes N"] \arrow[dr, "\lambda_N"] & E\otimes N \arrow[d, "\kappa"] \\
& & N \arrow[d, "r"] \\
X \arrow[urr, "\iota"] \arrow[rr, equal] & & X
\end{tikzcd}$$

The large left triangle commutes by naturality of λ. The top trapezoid commutes by functoriality of $-\otimes-$. The small middle right triangle commutes by unitality of κ. Finally, the bottom triangle commutes by definition, since we are assuming $r\circ\iota = \mathrm{id}_X$. Now the right composition is κ_X, so we have shown $\kappa_X\circ(e\otimes X) = \lambda_X$, as desired. Now, consider the following diagram:

$$\begin{tikzcd}
E\otimes E\otimes X \arrow[rrr, "\mu\otimes X"] \arrow[d, "E\otimes E\otimes\iota"'] \arrow[dr, "E\otimes E\otimes\iota"] & & & E\otimes X \arrow[d, "E\otimes\iota"] \\
E\otimes E\otimes N \arrow[r, "E\otimes E\otimes\ell"'] \arrow[d, "E\otimes\kappa"'] & E\otimes E\otimes N \arrow[rr, "\mu\otimes N"] \arrow[dd, "E\otimes\kappa"] & & E\otimes N \arrow[d, "\kappa"] \\
E\otimes N \arrow[d, "E\otimes r"'] \arrow[dr, "E\otimes\ell"] & & & N \arrow[d, "r"] \arrow[dl, equal] \\
E\otimes X \arrow[r, "E\otimes\iota"'] & E\otimes N \arrow[r, "\kappa"'] & N \arrow[r, "r"'] & X
\end{tikzcd}$$

The top trapezoid commutes by funtoriality of $-\otimes-$. The top left triangle commutes by functoriality of $-\otimes-$ and the fact that $\ell\circ\iota = \iota\circ r\circ\iota = \iota\circ\mathrm{id}_X = \iota$. The middle left trapezoid commutes by since ℓ is an E-module homomorphism, by assumption. The bottom left triangle commutes by functoriality of $- \otimes -$ and the fact that $\iota \circ r = \ell$. Thus, we have shown that (X, κ_X) is an E-module object, as desired.

Now, it remains to show that $\iota : X \to N$ and $r : N \to X$ are E-module homomorphisms. To that end, consider the following two diagrams:

$$\begin{tikzcd}
E \otimes X \arrow[r, "E\otimes\iota"] \arrow[d, "E\otimes\iota"'] & E \otimes N \arrow[ddd, "\kappa"] \\
E \otimes N \arrow[ur, "E\otimes\ell"'] \arrow[d, "\kappa"'] & \\
N \arrow[d, "r"'] \arrow[dr, "\ell"] & \\
X \arrow[r, "\iota"'] & N
\end{tikzcd}
\qquad
\begin{tikzcd}
E \otimes N \arrow[r, "E\otimes r"] \arrow[dr, "E\otimes\ell"'] \arrow[ddd, "\kappa"'] & E \otimes X \arrow[d, "E\otimes\iota"] \\
 & E \otimes N \arrow[d, "\kappa"] \\
 & N \arrow[d, "r"] \\
N \arrow[ur, "\ell"] \arrow[r, "r"] & X
\end{tikzcd}$$

The trapezoids in each diagram commute since we are assuming ℓ is a E-module homomorphism. The four triangles commute since $\ell\circ\iota = \iota$ and $r\circ\ell = r$. Thus, we have shown that $\kappa_X\circ(E\otimes r) = r\circ\kappa$ and $\kappa \circ (E \otimes \iota) = \iota \circ \kappa_X$, so we indeed have that ι and r are E-module homomorphisms, as desired. $\square$

Proposition C.14. *Suppose that $\mathcal{C}$ is an additive symmetric monoidal closed category. Let (E, μ, e) be a monoid object in $\mathcal{C}$, and suppose we have a family of E-module objects (N_i, κ_i) indexed by some small set I. Then $N := \bigoplus_{i\in I} N_i$ is canonically an E-module, with action map given by the composition*

$$\kappa : E \otimes \bigoplus_i N_i \xrightarrow{\cong} \bigoplus_i (E \otimes N_i) \xrightarrow{\bigoplus_i \kappa_i} \bigoplus_i N_i,$$

*where the first isomorphism is given by the fact that $E \otimes -$ preserves coproducts, since it is a left adjoint. Furthermore, N is the coproduct of all the N_i's in E-**Mod**, so that E-**Mod** has arbitrary coproducts.*

Proof. We need to show the action map κ makes the diagrams in Definition C.8 commute. To see the first (unitality) diagram commutes, consider the following diagram:

$$\begin{tikzcd}
\bigoplus_i N_i \arrow[r, "e\otimes\bigoplus_i N_i"] \arrow[dr, "\bigoplus_i(e\otimes N_i)"'] \arrow[ddr, equal] & E \otimes \bigoplus_i N_i \arrow[d, "\cong"] \\
 & \bigoplus_i(E \otimes N_i) \arrow[d, "\bigoplus_i \kappa_i"] \\
 & \bigoplus_i N_i
\end{tikzcd}$$

The top triangle commutes since $E \otimes -$ preserves coproducts, as it is a left adjoint. The bottom triangle commutes by unitality of each of the κ_i's. To see the second coherence diagram commutes, consider the following diagram:

$$\begin{tikzcd}
E \otimes E \otimes \bigoplus_i N_i \arrow[rr, "\mu \oplus \bigoplus_i N_i"] \arrow[d, "E\otimes\cong"'] \arrow[dr, "\cong"] & & E \otimes \bigoplus_i N_i \arrow[d, "\cong"] \\
E \otimes \bigoplus_i (E \otimes N_i) \arrow[r, "\cong"] \arrow[d, "E\otimes\bigoplus_i \kappa_i"'] & \bigoplus_i (E \otimes E \otimes N_i) \arrow[r, "\bigoplus_i(\mu\otimes N_i)"] \arrow[d, "\bigoplus_i(E\otimes\kappa_i)"'] & \bigoplus_i (E \otimes N_i) \arrow[d, "\bigoplus_i \kappa_i"] \\
E \otimes \bigoplus_i N_i \arrow[r, "\cong"'] & \bigoplus_i (E \otimes N_i) \arrow[r, "\bigoplus_i \kappa_i"'] & \bigoplus_i N_i
\end{tikzcd}$$

The bottom right square commutes by coherence for the κ_i's. Every other region commutes since $-\otimes-$ preserves colimits in each variable. Thus $N = \bigoplus_i N_i$ is indeed an E-module object, as desired.

Now, we claim that (N, κ) is the coproduct of the (N_i, κ_i)'s in E-**Mod**. First, we need to show that the canonical maps $\iota_i : N_i \hookrightarrow N$ are morphisms in E-**Mod** for all $i \in I$. To see ι_i is a homomorphism of E-module objects, consider the following diagram:

$$\begin{tikzcd}
E \otimes N_i \arrow[r, "E\otimes\iota_i"] \arrow[dr, hook, "\iota_{E\otimes N_i}"] \arrow[dd, "\kappa_i"'] & E \otimes \bigoplus_i N_i \arrow[d, "\cong"] \\
& \bigoplus_i (E \otimes N_i) \arrow[d, "\bigoplus_i \kappa_i"] \\
N_i \arrow[r, hook, "\iota_i"'] & \bigoplus_i N_i
\end{tikzcd}$$

The top triangle commutes by additivity of $E \otimes -$. The bottom trapezoid commutes since, by univeral property of the coproduct, $\bigoplus_i \kappa_i$ is the unique arrow which makes the trapezoid commute for all $i \in I$. Now, it remains to show that given an E-module object (N', κ') and homomorphisms $f_i : N_i \to N'$ of E-module objects for all $i \in I$, that the unique arrow $f : N \to N'$ in $\mathcal{SH}$ satisfying $f \circ \iota_i = f_i$ for all $i \in I$ is a homomorphism of E-module objects, so that N is actually the coproduct of the N_i's. To see this, first let $h : \bigoplus_i (E \otimes N_i) \to E \otimes N'$ be the arrow determined by the maps $E \otimes N_i \xrightarrow{E\otimes f_i} E \otimes N'$. Then consider the following diagram:

$$\begin{tikzcd}
E \otimes \bigoplus_i N_i \arrow[rr, "E\otimes f"] \arrow[d, "\cong"'] & & E \otimes N' \arrow[ddd, "\kappa'"] \\
\bigoplus_i (E \otimes N_i) \arrow[r, "\bigoplus_i(E\otimes f_i)"'] \arrow[urr, "h"] \arrow[dd, "\bigoplus_i \kappa_i"'] & \bigoplus_i (E \otimes N') \arrow[ur, "\nabla"'] \arrow[d, "\bigoplus_i \kappa'"] & \\
& \bigoplus_i N' \arrow[dr, "\nabla"] & \\
\bigoplus_i N_i \arrow[ur, "\bigoplus_i f_i"] \arrow[rr, "f"'] & & N'
\end{tikzcd}$$

The top triangle commutes by additivity of $E \otimes -$. The triangle below that commutes by the universal property of the coproduct, since it is straightforward to check that $\nabla \circ \bigoplus_i (E \otimes f_i)$ and h both satisfy the universal property of the colimit. The left trapezoid commutes by functoriality of $-\oplus-$ and the fact that f_i is a homomorphism of E-module objects for all i in I. The right trapezoid commutes by naturality of ∇. Finally, the bottom triangle commutes by the universal product of the coproduct, by showing that $\nabla \circ \bigoplus_i f_i$ in place of f also satisfies the universal property of the colimit. Hence f is inded a homomorphism of E-module objects, as desired.

To recap, we have shown that given a set of E-module objects $\{(N_i, \kappa_i)\}_{i \in I}$, the inclusion maps $\iota_i : N_i \hookrightarrow \bigoplus_i N_i$ are morphisms in E-**Mod**, and that given morphisms $f_i : (N_i, \kappa_i) \to (N', \kappa')$ for all $i \in I$, the unique induced map $\bigoplus_i N_i \to N'$ is a morphism in E-**Mod**. Thus, E-**Mod** does indeed have arbitrary coproducts, and the forgetful functor E-**Mod** $\to \mathcal{SH}$ preserves them. $\square$

Proposition C.15. *Suppose that $\mathcal{C}$ is an additive closed symmetric monoidal category, and let (E, μ, e) be a monoid object in $\mathcal{C}$. Then E-**Mod** is itself an additive category, so that in particular the forgetful functor E-**Mod** $\to \mathcal{C}$ and the free functor $\mathcal{C} \to E$-**Mod** (Proposition C.12) are additive.*

Proof. It is a general fact that adjoint functors between additive categories are necessarily additive. In order to show E-**Mod** is an additive category, it suffices to show it has finite coproducts, that $\mathrm{Hom}_{E\text{-}\mathbf{Mod}}(N, N')$ is an abelian group for all E-modules N and N', and that composition is bilinear. We know that E-**Mod** has coproducts which are preserved by the forgetful functor E-**Mod** $\to \mathcal{C}$ by Proposition C.14 (which is clearly faithful). Thus, because $\mathcal{C}$ is **Ab**-enriched and $\mathrm{Hom}_{E\text{-}\mathbf{Mod}}(N, N') \subseteq \mathcal{C}(N, N')$, it suffices to show that $\mathrm{Hom}_{E\text{-}\mathbf{Mod}}(N, N')$ is closed under addition and taking inverses. To see the former, let $f, g : N \to N'$ be E-module homomorphisms, and consider the following diagram:

$$\begin{tikzcd}
E \otimes N \arrow[r, "E\otimes\Delta_N"] \arrow[dr, "\Delta_{E\otimes N}"'] \arrow[dd, "\kappa"'] & E \otimes (N \oplus N) \arrow[r, "E\otimes(f\oplus g)"] \arrow[d, "\cong"'] & E \otimes (N' \oplus N') \arrow[r, "E\otimes\nabla_{N'}"] \arrow[d, "\cong"] & E \otimes N' \arrow[dd, "\kappa'"] \\
& (E \otimes N) \oplus (E \otimes N) \arrow[r, "(E\otimes f)\oplus(E\otimes g)"] \arrow[d, "\kappa\oplus\kappa"'] & (E \otimes N') \otimes (E \otimes N') \arrow[ur, "\nabla_{E\otimes N'}"'] \arrow[d, "\kappa'\oplus\kappa'"] & \\
N \arrow[r, "\Delta_N"] & N \oplus N \arrow[r, "f\oplus g"] & N' \oplus N' \arrow[r, "\nabla_{N'}"] & N'
\end{tikzcd}$$

The outermost trapezoids commute by naturality of Δ and ∇. The triangles in the top corners and the top middle rectangle commute by additivity of $E \otimes -$. Finally, the middle bottom rectangle commutes by functoriality of $- \oplus -$ and $- \otimes -$, and the fact that f and g are E-module homomorphisms. Commutativity of the above diagram shows that $f + g$ is a homomorphism of E-modules as desired. Finally, to see $-f$ is a E-module homomorphism if f is, we would like to show that $\kappa' \circ (E \otimes (-f)) = (-f) \circ \kappa$. This follows by the fact that $\kappa' \circ (E \otimes f) = f \circ \kappa$ and additivity of $- \otimes -$ and composition. $\square$

Appendix D. Homological (co)algebra

The primary reference for this section will be the nLab page on derived functors in homological algebra ([21]).

Recall that given abelian categories $\mathcal{A}$ and $\mathcal{B}$, given an additive functor $F : \mathcal{A} \to \mathcal{B}$, if F is left exact and $\mathcal{A}$ has enough injectives, we may form the *right derived functors* $R^n F : \mathcal{A} \to \mathcal{B}$ of F, for $n \in \mathbb{N}$. Given an object A in $\mathcal{A}$, we may compute $R^n F(A)$ to be the object (defined only up to isomorphism) which is obtained as follows: First, fix an injective resolution $i : A \to I^*$ of A, i.e., the data of a long exact sequence

$$0 \longrightarrow A \xrightarrow{i} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \xrightarrow{d^2} I^3 \longrightarrow \cdots$$

where each I^n is an injective object in $\mathcal{A}$. Such a sequence is guaranteed to exist since $\mathcal{A}$ has enough injectives. Then we define $R^n F(A)$ to be the n^{th} cohomology group $H^n(F(I^*))$ of the sequence

$$0 \longrightarrow F(I^0) \xrightarrow{F(d^0)} F(I^1) \xrightarrow{F(d^1)} F(I^2) \xrightarrow{F(d^2)} F(I^3) \longrightarrow \cdots .$$

It is a standard result that this definition of $R^n F(A)$ does not depend on the choice of injective resolution $i : A \to I^*$.

Definition D.1. Given an abelian category $\mathcal{A}$ with enough injectives and an object A in $\mathcal{A}$, we denote the right derived functors of the left exact functor $\mathrm{Hom}_{\mathcal{A}}(A,-) : \mathcal{A} \to \mathbf{Ab}$ by

$$\mathrm{Ext}^n_{\mathcal{A}}(A,-) := R^n\mathrm{Hom}_{\mathcal{A}}(A,-).$$

Remark D.2. It is not uncommon to instead define $\mathrm{Ext}^n_{\mathcal{A}}(-,A)$ to be the right derived functor of the functor $\mathrm{Hom}_{\mathcal{A}}(-,A) : \mathcal{A}^{\mathrm{op}} \to \mathbf{Ab}$, in which case we may compute $\mathrm{Ext}^n_{\mathcal{A}}(B,A)$ by means of *projective* resolutions of A in $\mathcal{A}$. It is a standard result that these definitions of $\mathrm{Ext}^n_{\mathcal{A}}(A,B)$ coincide.

Now, the first result we will state is that in order to compute the values of the right derived functors $R^nF(A)$, we do not need to consider strictly injective resolutions of A, rather, we may consider more generally "F-acyclic resolutions". First, we define F-acyclic objects:

Definition D.3 ([21, Definition 3.8])**.** Let $F : \mathcal{A} \to \mathcal{B}$ be a left or right exact additive functor between abelian categories, and suppose $\mathcal{A}$ has enough injectives. An object A in $\mathcal{A}$ is called an *F-acyclic object* if $R^nF(A) = 0$ for all $n > 0$.

Definition D.4. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact additive functor between abelian categories, and suppose $\mathcal{A}$ has enough injectives. Then given an object A in $\mathcal{A}$, an *F-acyclic resolution* $i : A \to I^*_F$ is the data of a long exact sequence in $\mathcal{A}$

$$0 \longrightarrow A \xrightarrow{i} I^0_F \xrightarrow{d^0} I^1_F \xrightarrow{d^1} I^2_F \xrightarrow{d^2} I^3_F \longrightarrow \cdots$$

such that each I^n_F is an F-acyclic object in $\mathcal{A}$.

The reason that F-acyclic objects are useful is that they allow you to compute the right derived functors of F without having to use strictly injective resolutions:

Proposition D.5 ([21, Theorem 3.15])**.** *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact additive functor between abelian categories. Then for each object A in $\mathcal{A}$, given an F-acyclic resolution $i : A \to I^*_F$ of A, for each $n \in \mathbb{N}$ there is a canonical isomorphism*

$$R^nF(A) \cong H^n(F(I^*_F))$$

between the n^{th} right derived functor of F evaluated on A and the cohomology of the sequence obtained by applying F to I^*_F.

Appendix E. Hopf algebroids

In this appendix, we will define the notion of *A-graded anticommutative Hopf algebroids* (Definition E.2) over an A-graded anticommutative ring R (Definition 4.5), and left comodules over them (Definition E.6).

E.1. A-graded anticommutative Hopf algebroids over R.

Given an A-graded anticommutative ring R, we will define an A-graded anticommutative Hopf algebroid over R to be a cogroupoid object in R-$\mathbf{GCA}^A$, i.e., a groupoid object in $(R\text{-}\mathbf{GCA}^A)^{\mathrm{op}}$. First, recall the definition of a *groupoid object* in a category with pullbacks:

Definition E.1. Let $\mathcal{C}$ be a category with pullbacks. A *groupoid object* in $\mathcal{C}$ consists of a pair of objects (M,O) together with five morphisms

(1) *Source and target*: $s, t : M \to O$,
(2) *Identity*: $e : O \to M$,
(3) *Composition*: $c : M \times_O M \to M$,
(4) *Inverse*: $i : M \to M$

Where $M \times_O M$ will always refer to the object which into the following pullback diagram in $\mathcal{C}$:

$$\begin{tikzcd}
M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, "t"] \\
M \arrow[r, "s"'] & O
\end{tikzcd}$$

For example, if we're working in $\mathcal{C} = \mathbf{Set}$, we should think of M as a set of morphisms, and O as a set of objects. The functions s and t take a morphism to their domain and codomain, respectively, and $M \times_O M$ is the collection of pairs of morphisms $(g, f) \in M \times M$ such that $t(f) = s(g)$, and the composition map $c : M \times_O M \to M$ takes such a pair to the element $g \circ f \in M$. We think of the identity $e : O \to M$ as taking some object $x \in O$ to the identity morphism $e(x) = \mathrm{id}_x \in M$ on x, and the inverse map $i : M \to M$ takes a morphism f to its inverse f^{-1}. These data are required to make the following diagrams commute:

(1) Composition works correctly:

$$\begin{tikzcd}
M \times_O M \arrow[r, "c"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\
M \arrow[r, "t"] & O
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[dr, "s"'] & O \arrow[l, "e"'] \arrow[r, "e"] \arrow[d, equal] & M \arrow[dl, "t"] \\
& O &
\end{tikzcd}
\qquad
\begin{tikzcd}
M \times_O M \arrow[r, "p_2"] \arrow[d, "c"'] & M \arrow[d, "s"] \\
M \arrow[r, "s"] & O
\end{tikzcd}$$

Expressed in terms of sets, the first diagram says that the target of $g \circ f$ is the target of g. The second diagram says that the domain and codomain of the identity on some object x is x. The third diagram says that the domain of $g \circ f$ is the domain of f.

(2) Associativity of composition: Write $M \times_O (M \times_O M)$ and $(M \times_O M) \times_O M$ for the pullbacks of $(s, t \circ c)$ and $(s \circ c, t)$, respectively, so we have commuting diagrams

$$\begin{tikzcd}
(M \times_O M) \times_O M \arrow[rr, "p_2'"] \arrow[dr, dashed, "c \times M"'] \arrow[dd, "p_1'"'] & & M \arrow[d, equal] \\
& M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\
M \times_O M \arrow[r, "c"] & M \arrow[r, "s"] & O
\end{tikzcd}
\qquad
\begin{tikzcd}
M \times_O (M \times_O M) \arrow[rr, "p_2''"] \arrow[dr, dashed, "M \times c"'] \arrow[dd, "p_1''"'] & & M \times_O M \arrow[d, "c"] \\
& M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\
M \arrow[r, equal] & M \arrow[r, "s"] & O
\end{tikzcd}$$

where the inner and outer squares in both diagrams are pullback squares. Furthermore, assuming the diagrams in condition (1) above are satisfied, we have that $t \circ p_1 \circ p_2'' = t \circ c \circ p_2'' = s \circ p_1''$, so that by the universal property of the pullback we have a map $M \times p_1 : M \times_O (M \times_O M) \to M \times_O M$ like so:

$$\begin{tikzcd}
M \times_O (M \times_O M) \arrow[drr, bend left, "p_1 \circ p_2''"] \arrow[dr, dashed, "M \times p_1"] \arrow[ddr, bend right, "p_1''"'] & & \\
& M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\
& M \arrow[r, "s"] & O
\end{tikzcd}$$

Now note that again assuming the diagrams above in (1) commute, we have $s \circ c = s \circ p_2$, so that

$$s \circ c \circ (M \times p_1) = s \circ p_2 \circ (M \times p_1) = s \circ p_1 \circ p_2'' = t \circ p_2 \circ p_2''.$$

Then by the unviersal property of the pullback we get a map $a : M \times_O (M \times_O M) \to (M \times_O M) \times_O M$ like so:

$$\begin{tikzcd}
M \times_O (M \times_O M) \arrow[dr, dashed, "a"] \arrow[drrr, bend left, "p_2 \circ p_2''"] \arrow[dddr, bend right, "M \times p_1"'] & & & \\
& (M \times_O M) \times_O M \arrow[rr, "p_2'"] \arrow[dd, "p_1'"'] & & M \arrow[d, equal] \\
& & & M \arrow[d, "t"] \\
& M \times_O M \arrow[r, "c"] & M \arrow[r, "s"] & O
\end{tikzcd}$$

Exercise: Show that this map a is an isomorphism. Then we require that the following diagram commutes:

$$\begin{tikzcd}
M \times_O (M \times_O M) \arrow[rr, "a"] \arrow[d, "M \times c"'] & & (M \times_O M) \times_O M \arrow[d, "c \times M"] \\
M \times_O M \arrow[r, "c"] & M & M \times_O M \arrow[l, "c"']
\end{tikzcd}$$

Expressed in terms of sets, this diagram says $h \circ (g \circ f) = (h \circ g) \circ f$.

(3) Unitality of composition: Given the maps $(\mathrm{id}_M, e \circ t), (e \circ s, \mathrm{id}_M) : M \to M \times_O M$ defined by the universal property of $M \times_O M$:

$$\begin{tikzcd}
M \arrow[dr, dashed, "(\mathrm{id}_M, e\circ s)"] \arrow[drr, bend left, "e \circ s"] \arrow[ddr, bend right, equal] & & \\
& M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, "t"] \\
& M \arrow[r, "s"] & O
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[dr, dashed, "(e\circ t, \mathrm{id}_M)"] \arrow[drr, bend left, equal] \arrow[ddr, bend right, "e \circ t"'] & & \\
& M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, "t"] \\
& M \arrow[r, "s"] & O
\end{tikzcd}$$

the following diagram commutes:

$$\begin{tikzcd}
M \arrow[r, "(e\circ t, \mathrm{id}_M)"] \arrow[d, "(\mathrm{id}_M, e\circ s)"'] \arrow[dr, equal] & M \times_O M \arrow[d, "c"] \\
M \times_O M \arrow[r, "c"'] & M
\end{tikzcd}$$

Expressed in terms of sets, this diagram says that given $f \in M$ with $s(f) = x$ and $t(f) = y$, that $f \circ \mathrm{id}_x = f$ and $\mathrm{id}_y \circ f = f$.

(4) Inverse: The following diagrams must commute:

$$\begin{tikzcd}
& M \arrow[d, "i"] \arrow[dl, equal] \\
M & M \arrow[l, "i"']
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[r, "(\mathrm{id}_M, i)"] \arrow[d, "t"'] & M \times_O M \arrow[d, "c"] & M \arrow[l, "(i, \mathrm{id}_M)"'] \arrow[d, "s"] \\
O \arrow[r, "e"] & M & O \arrow[l, "e"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& M \arrow[dl, "s"'] \arrow[d, "i"] \arrow[dr, "t"] & \\
O & M \arrow[l, "t"'] \arrow[r, "s"] & O
\end{tikzcd}$$

where the arrows (id_M, i) and (i, id_M) are determined by the universal property of $M \times_O M$ like so:

$$\begin{tikzcd}
M \arrow[dr, dashed, "(\mathrm{id}_M, i)"] \arrow[drr, bend left, "i"] \arrow[ddr, bend right, equal] & & \\
& M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\
& M \arrow[r, "s"] & O
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[dr, dashed, "(i, \mathrm{id}_M)"] \arrow[drr, bend left, equal] \arrow[ddr, bend right, "i"'] & & \\
& M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\
& M \arrow[r, "s"] & O
\end{tikzcd}$$

Expressed in terms of sets, given $f \in M$ with $s(f) = x$ and $t(f) = y$, the first diagram says that $(f^{-1})^{-1} = f$, the second says that $f \circ f^{-1} = \mathrm{id}_y$ and $f^{-1} \circ f = \mathrm{id}_x$, and the last diagram says that the domain and codomain of f^{-1} are y and x, respectively.

It can be seen that groupoid objects in $\mathcal{C} = \mathbf{Set}$ are precisely (small) groupoids. Now, we can state and unravel the definition of a Hopf algebroid:

Definition E.2. Given an A-graded anticommutative ring R (Definition 4.5), an *A-graded anticommutative Hopf algebroid over R* is a co-groupoid object in R-$\mathbf{GCA}^A$, i.e., a groupoid object in $(R\text{-}A\mathbf{GrCAlg})^{\mathrm{op}}$. Explicitly, an A-graded anticommutative Hopf algebroid over E is a pair (Γ, B) of objects in R-$A\mathbf{GrCAlg}$ along with morphisms

(1) *left unit*: $\eta_L : B \to \Gamma$ (corresponding to t),
(2) *right unit*: $\eta_R : B \to \Gamma$ (corresponding to s),
(3) *comultiplication*: $\Psi : \Gamma \to \Gamma \otimes_B \Gamma$ (corresponding to c),
(4) *counit*: $\epsilon : \Gamma \to B$ (corresponding to e),
(5) *conjugation*: $c : \Gamma \to \Gamma$ (corresponding to i),

where here Γ may be viewed as a B-bimodule with left B-module structure induced by η_L and right B-module structure induced by η_R, so we may form the tensor product of bimodules $\Gamma \otimes_B \Gamma$, which further may be given the structure of an A-graded anticommutative R-algebra (by Proposition B.21), and fits into the following pushout diagram in R-$\mathbf{GCA}^A g$ (Proposition B.22):

$$\begin{tikzcd}
B \arrow[r, "\eta_L"] \arrow[d, "\eta_R"'] & \Gamma \arrow[d, "g \mapsto 1 \otimes g"] \\
\Gamma \arrow[r, "g \mapsto g \otimes 1"'] & \Gamma \otimes_B \Gamma
\end{tikzcd}$$

These data must make the following diagrams commute:

(1) (Composition works correctly)

$$\begin{tikzcd}
B \arrow[r, "\eta_L"] \arrow[d, "\eta_L"'] & \Gamma \arrow[d, "\Psi"] \\
\Gamma \arrow[r, "g \mapsto g \otimes 1"'] & \Gamma \otimes_B \Gamma
\end{tikzcd}
\qquad
\begin{tikzcd}
& B \arrow[dl, "\eta_R"'] \arrow[d, equal] \arrow[dr, "\eta_L"] & \\
\Gamma \arrow[r, "\epsilon"'] & B & \Gamma \arrow[l, "\epsilon"]
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "\eta_R"] \arrow[d, "\eta_R"'] & \Gamma \arrow[d, "g \mapsto 1 \otimes g"] \\
\Gamma \arrow[r, "\Psi"] & \Gamma \otimes_B \Gamma
\end{tikzcd}$$

(2) (Coassociativity) The following diagram must commute

$$\begin{tikzcd}
\Gamma \otimes_B \Gamma \arrow[d, "\Psi \otimes_B \Gamma"'] & \Gamma \arrow[l, "\Psi"'] \arrow[r, "\Psi"] & \Gamma \otimes_B \Gamma \arrow[d, "\Gamma \otimes_B \Psi"] \\
(\Gamma \otimes_B \Gamma) \otimes_B \Gamma \arrow[rr, "\cong"] & & \Gamma \otimes_B (\Gamma \otimes_B \Gamma)
\end{tikzcd}$$

where $(\Gamma \otimes_B \Gamma) \otimes_B \Gamma$ and $\Gamma \otimes_B (\Gamma \otimes_B \Gamma)$ denote the rings which fit into the following pushout diagrams in R-$\mathbf{GCA}^A$:

$$\begin{tikzcd}
B \arrow[rr, "\eta_L"] \arrow[d, "\eta_R"'] & & \Gamma \arrow[dd, "g \mapsto (1\otimes 1)\otimes g"] \\
\Gamma \arrow[d, "\Psi"'] & & \\
\Gamma \otimes_B \Gamma \arrow[rr, "g\otimes g' \mapsto (g\otimes g')\otimes 1"'] & & (\Gamma \otimes_B \Gamma) \otimes_B \Gamma
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "\eta_L"] \arrow[d, "\eta_R"'] & \Gamma \arrow[r, "\Psi"] & \Gamma \otimes_B \Gamma \arrow[d, "(g\otimes g')\mapsto 1\otimes(g\otimes g')"] \\
\Gamma \arrow[rr, "g \mapsto g\otimes(1\otimes 1)"'] & & \Gamma \otimes_B (\Gamma \otimes_B \Gamma)
\end{tikzcd}$$

and the isomorphism $(\Gamma \otimes_B \Gamma) \otimes_B \Gamma \to \Gamma \otimes_B (\Gamma \otimes_B \Gamma)$ sends $(g \otimes g') \otimes g''$ to $g \otimes (g' \otimes g'')$, the left vertical arrow $\Psi \otimes \Gamma$ sends $g \otimes g'$ to $\Psi(g) \otimes g$, and the right vertical arrow $\Gamma \otimes \Psi$ sends $g \otimes g'$ to $g \otimes \Psi(g')$.

(3) (Co-unitality):

$$\begin{tikzcd}
\Gamma \arrow[r, "\Psi"] \arrow[d, "\Psi"'] \arrow[dr, equal] & \Gamma \otimes_B \Gamma \arrow[d, "(\eta_L\circ\epsilon)\cdot \mathrm{id}_\Gamma"] \\
\Gamma \otimes_B \Gamma \arrow[r, "\mathrm{id}_\Gamma\cdot(\eta_R\circ\epsilon)"'] & \Gamma
\end{tikzcd}$$

where the right vertical arrow sends $g \otimes g'$ to $\eta_L(\epsilon(g))g'$ and the bottom horizontal arrow sends $g \otimes g'$ to $g\eta_R(\epsilon(g'))$.

(4) (Convolution):

$$\begin{tikzcd}
 & \Gamma \arrow[d, "c"] \arrow[dl, equal] \\
\Gamma & \Gamma \arrow[l, "c"]
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[d, "\eta_L"'] & \Gamma \arrow[l, "\epsilon"'] \arrow[r, "\epsilon"] \arrow[d, "i"'] & B \arrow[d, "\eta_R"] \\
\Gamma & \Gamma \otimes_B \Gamma \arrow[l, "\mathrm{id}_\Gamma\cdot c"] \arrow[r, "c\cdot\mathrm{id}_\Gamma"'] & \Gamma
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "\eta_L"] \arrow[dr, "\eta_R"'] & \Gamma \arrow[d, "c"'] & B \arrow[l, "\eta_R"'] \arrow[dl, "\eta_L"] \\
 & \Gamma &
\end{tikzcd}$$

where the bottom left arrow in the middle diagram sends $g \otimes g'$ to $gc(g')$ and the bottom right arrow in the middle diagram sends $g \otimes g'$ to $c(g)g'$.

The remainder of this subsection is devoted to proving some technical lemmas about A-graded anticommutative Hopf algebroids.

Proposition E.3. *Suppose we have an A-graded anticommutative Hopf algebroid (Γ, B) over (R, θ) with structure maps η_L, η_R, Ψ, ϵ, and c (Definition E.2). Recall in the definition, we considered $\Gamma \otimes_B \Gamma$ to be the A-graded R-commutative ring whose underlying abelian group was given by the tensor product of B-bimodules, where Γ has left B-module structure induced by η_L and right B-module structure induced by η_R. Thus $\Gamma \otimes_B \Gamma$ is canonically a B-bimodule, as it is a tensor product of B-bimodules. Then the canonical left (resp. right) B-module structure on $\Gamma \otimes_B \Gamma$ coincides with that induced by the ring homomorphism $\Psi \circ \eta_L$ (resp. $\Psi \circ \eta_R$).*

Proof. First we show the left module structures coincide. By additivity, in order to show the module structures coincide, it suffices to show that given a homogeneous pure tensor $g \otimes g'$ in $\Gamma \otimes_B \Gamma$ and some $b \in B$ that $\Psi(\eta_L(b)) \cdot (g \otimes g') = (\eta_L(b) \cdot g) \otimes g'$, where $\cdot$ on the left denotes the product in $\Gamma \otimes_B \Gamma$ and the $\cdot$ on the right denotes the product in Γ. By the axioms for a Hopf algebroid, we have that $\Psi(\eta_L(b)) = \eta_L(b) \otimes 1$. Thus by how the product in $\Gamma \otimes_B \Gamma$ is defined (Proposition B.21), we have that

$$\Psi(\eta_L(b)) \cdot (g \otimes g') = (\eta_L(b) \otimes 1) \cdot (g \otimes g') = (\varphi_\Gamma(\theta_{0,|g|}) \cdot \eta_L(b) \cdot g) \otimes (g' \cdot 1) = (\eta_L(b) \cdot g) \otimes g',$$

where $\varphi_\Gamma : R \to \Gamma$ is the structure map, and the last equality follows by the fact that $\theta_{0,|g|} = 1$. An entirely analagous argument yields that the canonical right module structure on $\Gamma \otimes_B \Gamma$ coincides with that induced by $\Psi \circ \eta_R$, since $\Psi \circ \eta_R = 1 \otimes \eta_R$. $\square$

Remark E.4. By the above proposition, given an A-graded commutative Hopf algebroid (Γ, B) over R, there is no ambiguity when discussing the objects $\Gamma \otimes_B (\Gamma \otimes_B \Gamma)$ and $(\Gamma \otimes_B \Gamma) \otimes_B \Gamma$ — they may both be considered as the threefold tensor product of the B-bimodule Γ with itself. In particular, we have a canonical isomorphism of B-bimodules

$$(\Gamma \otimes_B \Gamma) \otimes_B \Gamma \to \Gamma \otimes_B (\Gamma \otimes_B \Gamma)$$

sending $(g \otimes g') \otimes g''$ to $g \otimes (g' \otimes g'')$, and this is precisely the isomorphism in the coassociativity diagram in the definition of a Hopf algebroid (Definition E.2).

Proposition E.5. *Suppose we have an A-graded commutative Hopf algebroid (Γ, B) over R with structure maps η_L, η_R, Ψ, ϵ, and c. Then $\eta_L : B \to \Gamma$ is a homomorphism of left B-modules, $\eta_R : B \to \Gamma$ is a homomorphism of right B-modules, and $\Psi : \Gamma \to \Gamma \otimes_B \Gamma$ and $\epsilon : \Gamma \to B$ are homomorphisms of B-bimodules.*

Proof. Since the left (resp. right) B-module structure on Γ is induced by η_L (resp. η_R), the map η_L (resp. η_R) is a homomorphism of left (resp. right) B-modules by definition.

Next, we want to show Ψ is a homomorphism of B-bimodules. The left (resp. right) B-module structure on Γ is that induced by η_L (resp. η_R), and in Proposition E.3, we showed that the left (resp. right) B-module structure on $\Gamma \otimes_B \Gamma$ is that induced by $\Psi \circ \eta_L$ (resp. $\Psi \circ \eta_R$), so that by definition $\Psi : \Gamma \to \Gamma \otimes_B \Gamma$ is a homomorphism of left (resp. right) B-modules.

Lastly, we claim that $\epsilon : \Gamma \to B$ is a homomorphism of B-bimodules. We need to show that given $g \in \Gamma$ and $b, b' \in B$ that $\epsilon(\eta_L(b) g \eta_R(g')) = b\epsilon(g)b'$. This follows from the fact that ϵ is a ring homomorphism satisfying $\epsilon \circ \eta_L = \epsilon \circ \eta_R = \mathrm{id}_B$. $\square$

E.2. **Comodules over a Hopf algebroid.** In what follows, fix an A-graded anticommutative ring (R, θ) and an A-graded anticommutative Hopf algebroid (Γ, B) over R with structure maps η_L, η_R, Ψ, ϵ, and c. We will always view Γ with its *canonical* B-bimodule structure, with left B-module structure induced by η_L, and right B-module structure induced by η_R. In particular, any tensor product over B involving Γ will always refer to Γ with this bimodule structure.

Definition E.6. A *left comodule over* Γ is a pair (N, Ψ_N), where N is a left A-graded B-module and $\Psi_N : N \to \Gamma \otimes_B N$ is an A-graded homomorphism of left A-graded B-modules. These data are required to make the following diagrams commute

$$\begin{tikzcd} N \arrow[r, "\Psi_N"] \arrow[dr, "\cong"'] & \Gamma \otimes_B N \arrow[d, "\epsilon \otimes N"] \\ & B \otimes_B N \end{tikzcd} \qquad \begin{tikzcd} \Gamma \otimes_B N \arrow[d, "\Psi \otimes N"'] & N \arrow[l, "\Psi_N"'] \arrow[r, "\Psi_N"] & \Gamma \otimes_B N \arrow[d, "\Gamma \otimes \Psi_N"] \\ (\Gamma \otimes_B \Gamma) \otimes_B N \arrow[rr, "\cong"] & & \Gamma \otimes_B (\Gamma \otimes_B N) \end{tikzcd}$$

The maps $\epsilon \otimes N$ and $\Psi \otimes N$ are well-defined by Proposition E.5, and the bottom isomorphism in the right diagram is the canonical one sending $(g \otimes g') \otimes n \mapsto g \otimes (g' \otimes n)$.

Given two left A-graded Γ-comodules (N_1, Ψ_{N_1}) and (N_2, Ψ_{N_2}), a homomorphism of left A-graded comodules $f : N_1 \to N_2$ is an A-graded homomorphism of the underlying left B-modules such that the following diagram commutes:

$$\begin{tikzcd} N_1 \arrow[r, "f"] \arrow[d, "\Psi_{N_1}"'] & N_2 \arrow[d, "\Psi_{N_2}"] \\ \Gamma \otimes_B N_1 \arrow[r, "\Gamma \otimes f"] & \Gamma \otimes_B N_2 \end{tikzcd}$$

We write Γ-$\mathbf{CoMod}^A$ for the resulting category of left A-graded comodules over Γ. In the above definition, we required A-graded left Γ-comodule homomorphisms to strictly preserve the grading, but we could have instead considered left Γ-comodule homomorphisms which are of degree d for some $d \in A$, or equivalently, the set of degree zero A-graded Γ-comodule homomorphisms from N_1 to the shifted comodule $(N_2)_{*+d}$. We denote the hom-set of degree-d A-graded left Γ-comodule homomorphisms from (N_1, Ψ_{N_1}) to (N_2, Ψ_{N_2}) by

$$\mathrm{Hom}^d_{\Gamma\text{-}\mathbf{CoMod}^A}(N_1, N_2) \qquad \text{or usually just} \qquad \mathrm{Hom}^d_\Gamma(N_1, N_2).$$

In particular, write $\mathrm{Hom}_{\Gamma\text{-}\mathbf{CoMod}^A}(N_1, N_2)$ or just $\mathrm{Hom}_\Gamma(N_1, N_2)$ to mean the set of strictly degree preserving (degree 0) A-graded left Γ-comodule homomorphisms from (N_1, Ψ_{N_1}) to (N_2, Ψ_{N_2}).

Proposition E.7. *The category* Γ-$\mathbf{CoMod}^A$ *is an additive category.*

Proof. First, we show the category is $\mathbf{Ab}$-enriched. Since the forgetful functor Γ-$\mathbf{CoMod}^A \to B$-$\mathbf{Mod}^A$ is clearly faithful, we may view hom-sets in Γ-$\mathbf{CoMod}^A$ as subsets of hom-groups in B-$\mathbf{Mod}^A$, so that in order to show Γ-$\mathbf{CoMod}^A$ is $\mathbf{Ab}$-enriched, it suffices to show that hom-sets in Γ-$\mathbf{CoMod}^A$ are closed under addition of module homomorphisms and taking inverses. To that end, suppose we have two A-graded left Γ-comodule homomorphisms $f, g : (N_1, \Psi_{N_1}) \to (N_2, \Psi_{N_2})$, then we have

$$\begin{aligned}\Psi_{N_2} \circ (f + g) &= (\Psi_{N_2} \circ f) + (\Psi_{N_2} \circ g)\\ &= ((\Gamma \otimes_B f) \circ \Psi_{N_1}) + ((\Gamma \otimes_B g) \circ \Psi_{N_1})\\ &= ((\Gamma \otimes_B f) + (\Gamma \otimes_B g)) \circ \Psi_{N_1}\\ &= (\Gamma \otimes_B (f + g)) \circ \Psi_{N_1},\end{aligned}$$

where the first equality follows since Ψ_{N_2} is a homomorphism, the second follows since f and g are left Γ-comodule homomorphisms, the third follows since Ψ_{N_1} is a homomorphism, and the last equality follows by definition of the tensor product of modules. Hence $f + g$ is indeed an A-graded left Γ-comodule homomorphism, as desired. Now, we also claim $-f$ is an A-graded left Γ-comodule homomorphism. To that end, note that

$$\Psi_{N_2} \circ (-f) = -\Psi_{N_2} \circ f = -(\Gamma \otimes_B f) \circ \Psi_{N_1} = (\Gamma \otimes_B (-f)) \circ \Psi_{N_1},$$

where the first equality follows since Ψ_{N_2} is a homomorphism, the second follows since f is an A-graded left Γ-comodule homomorphism, and the third equality follows by definition of the tensor product.

Thus, we've shown that the hom-sets in Γ-$\mathbf{CoMod}^A$ are abelian groups, and composition is clearly bilinear, so that Γ-$\mathbf{CoMod}^A$ is indeed $\mathbf{Ab}$-enriched.

Now, in order to show Γ-$\mathbf{CoMod}^A$ is additive, it suffices to show that it contains a zero object and has binary coproducts. First of all, it is straightforward to check that the zero left B-module is clearly an A-graded left Γ-comodule with structure map the unique map $0 \to \Gamma \otimes_B 0 \cong 0$, and that given any other A-graded left Γ-comodule (N, Ψ_N), the unique homomorphisms of left B-modules $0 \to N$ and $N \to 0$ are left comodule homomorphisms.

Now, suppose we have two A-graded left Γ-comodules (N_1, Ψ_{N_1}) and (N_2, Ψ_{N_2}). First, we claim their direct sum as left B-modules $N_1 \oplus N_2$ is canonically an A-graded left Γ-comodule. We know that $N_1 \oplus N_2$ is an A-graded left B-module by Lemma B.11, and we can define the structure map

$$\Psi_{N_1 \oplus N_2} : N_1 \oplus N_2 \xrightarrow{\Psi_{N_1} \oplus \Psi_{N_2}} (\Gamma \otimes_B N_1) \oplus (\Gamma \otimes_B N_2) \cong \Gamma \otimes_B (N_1 \oplus N_2),$$

where the final isomorphism is the canonical one sending $(g_1 \otimes n_1) \oplus (g_2 \otimes n_2)$ to $(g_1 \otimes n_1) + (g_2 \otimes n_2)$. Then to see this is in fact a left Γ-comodule, first consider the following diagram:

$$\begin{tikzcd}
N_1 \oplus N_2 \arrow[r, "\Psi_{N_1} \oplus \Psi_{N_2}"] \arrow[dr, "\cong"] \arrow[ddrr, bend right] & (\Gamma \otimes_B N_1) \oplus (\Gamma \otimes_B N_2) \arrow[r, "\cong"] \arrow[d, dashed, "(\epsilon \otimes N_1) \oplus (\epsilon \otimes N_2)"] & \Gamma \otimes_B (N_1 \oplus N_2) \arrow[dd, "\epsilon \otimes (N_1 \oplus N_2)"] \\
 & (B \otimes_B N_1) \oplus (B \otimes_B N_2) \arrow[dr, "\cong"] & \\
 & & B \otimes_B (N_1 \oplus N_2)
\end{tikzcd}$$

A simple diagram chase yields the left and rightmost regions commute. The top left region commutes since (N_1, Ψ_{N_1}) and (N_2, Ψ_{N_2}) are left Γ-comodules. Now, consider the following diagram:

$$\begin{tikzcd}
\Gamma \otimes_B (N_1 \oplus N_2) \arrow[r, "\Gamma \otimes_B (\Psi_{N_1} \oplus \Psi_{N_2})"] & \Gamma \otimes_B ((\Gamma \otimes_B N_1) \oplus (\Gamma \otimes_B N_2)) \arrow[r, "\Gamma \otimes_B \cong"] & \Gamma \otimes_B (\Gamma \otimes_B (N_1 \oplus N_2)) \\
(\Gamma \otimes_B N_1) \oplus (\Gamma \otimes_B N_2) \arrow[u, "\cong"] \arrow[r, "(\Gamma \otimes \Psi_{N_1}) \oplus (\Gamma \otimes \Psi_{N_2})"] & (\Gamma \otimes_B (\Gamma \otimes_B N_1)) \oplus (\Gamma \otimes_B (\Gamma \otimes_B N_2)) \arrow[u, "\cong"] \arrow[dd, "\cong \oplus \cong"] & \\
N_1 \oplus N_2 \arrow[u, "\Psi_{N_1} \oplus \Psi_{N_2}"] \arrow[d, "\Psi_{N_1} \oplus \Psi_{N_2}"] & & \\
(\Gamma \otimes_B N_1) \oplus (\Gamma \otimes_B N_2) \arrow[r, "(\Psi \otimes N_1) \oplus (\Psi \otimes N_2)"] \arrow[d, "\cong"] & ((\Gamma \otimes_B \Gamma) \otimes_B N_1) \oplus ((\Gamma \otimes_B \Gamma) \otimes_B N_2) \arrow[dr, "\cong"] & \\
\Gamma \otimes_B (N_1 \oplus N_2) \arrow[rr, "\Psi \otimes (N_1 \oplus N_2)"'] & & (\Gamma \otimes_B \Gamma) \otimes_B (N_1 \oplus N_2) \arrow[uuuu, "\cong"]
\end{tikzcd}$$

The middle left region commutes since (N_1, Ψ_{N_1}) and (N_2, Ψ_{N_2}) are left Γ-comodules. Each other region in the diagram can be seen to commute by a straightforward diagram chase.

Thus, we have shown that $N_1 \oplus N_2$ is indeed canonically an A-graded left Γ-comodule. Then it remains to show that the canonical inclusions $\iota_i : N_i \hookrightarrow N_1 \oplus N_2$ are Γ-comodule homomorphisms for $i = 1, 2$, and that given Γ-comodule homomorphisms $(N_1, \Psi_{N_1}) \to (N, \Psi_N)$ and $(N_2, \Psi_{N_2}) \to (N, \Psi_N)$, that the map $N_1 \oplus N_2 \to N$ induced by the universal property of the coproduct in B-$\mathbf{Mod}^A$ is a Γ-comodule homomorphism. This is all entirely straightforward to check by doing a few simple diagram chases. $\square$

Proposition E.8. *The forgetful functor Γ-$\mathbf{CoMod}^A \to B$-$\mathbf{Mod}^A$ (where here B-$\mathbf{Mod}^A$ is the category of A-graded left B-modules and degree-preserving module homomorphisms between them) has a right adjoint $\Gamma \otimes_B - : B$-$\mathbf{Mod}^A \to \Gamma$-$\mathbf{CoMod}^A$ called the* co-free construction, *where the co-free left A-graded Γ-comodule on a left A-graded B-module M is the B-module $\Gamma \otimes_B M$ equipped with the coaction*

$$\Psi_{\Gamma \otimes_B M} : \Gamma \otimes_B M \xrightarrow{\Psi \otimes_B M} (\Gamma \otimes_B \Gamma) \otimes_B M \xrightarrow{\cong} \Gamma \otimes_B (\Gamma \otimes_B M).$$

Explicitly, given some (N, Ψ_N) in Γ-$\mathbf{CoMod}$ and some M in B-$\mathbf{Mod}^A$, the counit and unit of this adjunction are given by

$$\eta_{(N, \Psi_N)} : N \xrightarrow{\Psi_N} \Gamma \otimes_B N$$

and

$$\varepsilon_M : \Gamma \otimes_B M \xrightarrow{\epsilon \otimes_B M} B \otimes_B M \xrightarrow{\cong} M.$$

Proof. First, we need to show that given a left A-graded B-module that the given map $\Psi_{\Gamma\otimes_B M} : \Gamma \otimes_B M \to \Gamma \otimes_B (\Gamma \otimes_B M)$ endows B with the structure of a left Γ-comodule. To that end, first consider the following diagram:

$$\begin{tikzcd}
\Gamma \otimes_B M \arrow[r, "\Psi\otimes M"] \arrow[dr, equal] & (\Gamma \otimes_B \Gamma) \otimes_B M \arrow[rr, "\cong"] \arrow[d, "((\eta_L\circ\epsilon)\cdot \mathrm{id}_\Gamma)\otimes M"'] \arrow[dr, "(\epsilon\otimes\Gamma)\otimes M"] & & \Gamma \otimes_B (\Gamma \otimes_B M) \arrow[dd, "\epsilon\otimes(\Gamma\otimes M)"] \\
& \Gamma \otimes_B M \arrow[drr, "\cong"'] & (B \otimes_B \Gamma) \otimes_B M \arrow[l, "\eta_L\cdot \mathrm{id}_\Gamma"'] \arrow[dr, "\cong"] & \\
& & & B \otimes_B (\Gamma \otimes_B M)
\end{tikzcd}$$

The top left region commutes by the co-unitality axiom for a Hopf algebroid. A simple diagram chase yields commutativity of every other diagram (in particular, the bottom region commutes since the left B-module structure on Γ is that induced by η_L). Now, consider the following diagram:

$$\begin{tikzcd}
\Gamma \otimes_B (\Gamma \otimes_B M) \arrow[rr, "\Gamma\otimes(\Psi\otimes M)"] & & \Gamma \otimes_B (\Gamma \otimes_B (\Gamma \otimes_B M)) \arrow[dddd, "\Gamma\otimes\cong"] \\
(\Gamma \otimes_B \Gamma) \otimes_B M \arrow[u, "\cong"] \arrow[r, "(\Gamma\otimes\Psi)\otimes M"] & (\Gamma \otimes_B (\Gamma \otimes_B \Gamma)) \otimes_B M \arrow[ur, "\cong"] & \\
\Gamma \otimes_B M \arrow[u, "\Psi\otimes M"] \arrow[d, "\Psi\otimes M"'] & & \\
(\Gamma \otimes_B \Gamma) \otimes_B M \arrow[r, "(\Psi\otimes\Gamma)\otimes M"] \arrow[d, "\cong"'] & ((\Gamma \otimes_B \Gamma) \otimes_B \Gamma) \otimes_B M \arrow[uu, "\cong"'] & \\
\Gamma \otimes_B (\Gamma \otimes_B M) \arrow[r, "\Psi\otimes(\Gamma\otimes M)"] & (\Gamma \otimes_B \Gamma) \otimes_B (\Gamma \otimes_B M) \arrow[u, "\cong"] \arrow[r, "\cong"] & \Gamma \otimes_B ((\Gamma \otimes_B \Gamma) \otimes_B M)
\end{tikzcd}$$

The left region commutes since Ψ is co-associative. A simple diagram chase yields the commutativity of every other diagram. Thus, we have indeed shown that $(\Gamma \otimes_B M, \Psi_{\Gamma\otimes_B M})$ is an A-graded left Γ-comodule, as desired.

Now, we need to show that η and ε are natural transformations which satisy the zig-zag identities. The maps η is clearly natural by how morphisms in Γ-$\mathbf{CoMod}^A$ are defined. It is also clear that ε is natural by functoriality of $-\otimes_B -$. Thus, it remains to show the following two diagrams commute for all M in B-$\mathbf{Mod}^A$ and (N, Ψ_N) in Γ-$\mathbf{CoMod}^A$:

$$\begin{tikzcd}
N \arrow[r, "\eta_{(N,\Psi_N)}"] \arrow[dr, equal] & \Gamma \otimes_B N \arrow[d, "\varepsilon_N"] \\
& N
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma \otimes_B (\Gamma \otimes_B M) \arrow[d, "\Gamma\otimes_B \varepsilon_M"'] & \Gamma \otimes_B M \arrow[l, "\eta_{(\Gamma\otimes_B M, \Psi_{\Gamma\otimes_B M})}"'] \arrow[dl, equal] \\
\Gamma \otimes_B M &
\end{tikzcd}$$

Unravelling definitions, the left diagram becomes:

$$\begin{tikzcd}
N \arrow[r, "\Psi_N"] \arrow[dr, "\cong"] \arrow[ddr, equals] & \Gamma \otimes_B N \arrow[d, "\epsilon \otimes_B N"] \\
 & B \otimes_B N \arrow[d, "\cong"] \\
 & N
\end{tikzcd}$$

This commutes since (N, Ψ_N) is a left Γ-comodule. On the other hand, the right diagram becomes:

$$\begin{tikzcd}
\Gamma \otimes_B (\Gamma \otimes_B M) \arrow[d, "\Gamma \otimes (\epsilon \otimes M)"'] & (\Gamma \otimes_B \Gamma) \otimes_B M \arrow[l, "\cong"'] \arrow[d, "(\Gamma \otimes_B \epsilon) \otimes M"'] \arrow[ddr, "(\mathrm{id}_\Gamma \cdot (\eta_R \circ \epsilon)) \otimes M"] & \Gamma \otimes_B M \arrow[l, "\Psi \otimes M"'] \arrow[dd, equals] \\
\Gamma \otimes_B (B \otimes_B M) \arrow[d, "\Gamma \otimes \cong"'] & (\Gamma \otimes_B B) \otimes_B M \arrow[l, "\cong"'] \arrow[dl, "(\mathrm{id}_\Gamma \cdot \eta_R) \otimes M"'] & \\
\Gamma \otimes_B M \arrow[rr, equals] & & \Gamma \otimes_B M
\end{tikzcd}$$

The rightmost region commutes by co-unitality of Ψ, while a simple diagram chase yields commutativity of the remaining regions (in particular, the bottom let region commutes because the right B-module structure on Γ is induced by η_R). $\square$

Proposition E.9. *Suppose that Γ is flat as a right B-module, i.e., suppose $\eta_R : B \to \Gamma$ is a flat ring homomorphism. Then the category Γ-$\mathbf{CoMod}^A$ is an abelian category and has enough injectives.*

Proof. In Proposition E.7, we showed that Γ-$\mathbf{CoMod}^A$ is an additive category, so it remains to show that it has all kernels and cokernels, and that for all morphisms f in Γ-$\mathbf{CoMod}^A$ that the comparison morphism

$$\operatorname{coker}(\ker f) \to \ker(\operatorname{coker} f)$$

is an isomorphism. First, let $f : (N_1, \Psi_{N_1}) \to (N_2, \Psi_{N_2})$ be a morphism in Γ-$\mathbf{CoMod}^A$, and consider the following diagram:

$$\begin{tikzcd}
\ker f \arrow[r, hook] \arrow[d, dashed] & N_1 \arrow[r, "f"] \arrow[d, "\Psi_{N_1}"'] & N_2 \arrow[d, "\Psi_{N_2}"] \\
\Gamma \otimes_B \ker f \arrow[r, hook] & \Gamma \otimes_B N_1 \arrow[r, "\Gamma \otimes f"'] & \Gamma \otimes_B N_2
\end{tikzcd}$$

By the assumption that Γ is flat as a right B-module, we have that $\Gamma \otimes_B -$ is exact, so that in particular it preserves kernels, meaning $\Gamma \otimes_B \ker f = \ker(\Gamma \otimes_B f)$. This gives the bottom left horizontal arrow. Then by the universal property of the kernel in B-$\mathbf{Mod}^A$ and the fact that the right square commutes, we get the vertical dashed arrow which makes the left square commute, as desired, and that $\ker f$ with this structure map is indeed the kernel of f in Γ-$\mathbf{CoMod}$. Showing that this structure map makes the two diagrams in Definition E.6 commute is an exercise in diagram chasing and applying universal properties. Now, showing that the cokernel of f belongs to Γ-$\mathbf{CoMod}^A$ is formally dual. Finally, it follows from construction that the comparison morphism

$$\operatorname{coker}(\ker f) \to \ker(\operatorname{coker} f)$$

formed in Γ-$\mathbf{CoMod}^A$ is precisely the comparison morphism in B-$\mathbf{Mod}$, which is an isomorphism, and thus clearly an isomorphism in Γ-$\mathbf{CoMod}^A$ as well. Thus Γ-$\mathbf{CoMod}^A$ is indeed abelian, as desired. $\square$

Proposition E.10 ([25, Lemma 3.5])**.** *Suppose that Γ is flat as a right B-module, i.e., suppose $\eta_R : B \to \Gamma$ is a flat ring homomorphism. Let P be an A-graded left Γ-comodule in $\Gamma\text{-}\mathbf{CoMod}^A$ such that the underlying A-graded B-module is a graded projective module. Then every co-free module (Proposition E.8) is an F-acyclic object (Definition D.3) for the covariant hom functor* $\mathrm{Hom}_\Gamma(P, -)$.

Proof. We need to show that $\mathrm{Ext}^n_\Gamma(N, \Gamma \otimes_B M)$ vanishes for all A-graded B-modules M. First of all, let $i : M \to I^*$ be an injective resolution of M in $B\text{-}\mathbf{Mod}^A$, so we have an exact sequence of A-graded B-modules

$$0 \longrightarrow M \xrightarrow{i} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \xrightarrow{d^2} I^3 \longrightarrow \cdots .$$

Then Γ is flat as a right B-module, the sequence remains exact after we tensor it with Γ on the left. Furthermore, it is a general categorical fact that right adjoints between abelian catgories preserve injective objects. Thus $\Gamma \otimes i : \Gamma \otimes_B M \to \Gamma \otimes_B I^*$ is an injective resolution in $\Gamma\text{-}\mathbf{CoMod}^A$. Then for $n > 0$, we have

$$\mathrm{Ext}^n_\Gamma(N, \Gamma \otimes_B M) \cong H^n(\mathrm{Hom}_\Gamma(N, \Gamma \otimes_B I^*)) \cong H^n(\mathrm{Hom}_B(N, I^*)) \cong 0,$$

where the first isomorphism follows by the forgetful-cofree adjunction for comodules over a Hopf algebroid (Proposition E.8), and the final isomorphism follows by the fact that N is a graded projective module, i.e., a projective object in the abelian category $B\text{-}\mathbf{Mod}^A$, so that $\mathrm{Hom}_B(N, -)$ is an exact functor. $\square$

References

[1] J. F. Adams. "Lectures on generalised cohomology". In: *Category Theory, Homology Theory and Their Applications III*. Ed. by Peter J. Hilton. Berlin, Heidelberg: Springer Berlin Heidelberg, 1969, pp. 1–138. ISBN: 978-3-540-36140-4.

[2] J. F. Adams. "On the structure and applications of the steenrod algebra". In: *Commentarii Mathematici Helvetici* 32.1 (Dec. 1958), pp. 180–214. ISSN: 1420-8946. DOI: 10.1007/BF02564578. URL: https://doi.org/10.1007/BF02564578.

[3] Paul Balmer. *A guide to tensor-triangular classification*. 2019. arXiv: 1912.08963 [math.AT].

[4] Paul Balmer. "Tensor triangular geometry". In: *Proceedings of the International Congress of Mathematicians*. Vol. 2. 2010, pp. 85–112.

[5] J. Michael Boardman. "Conditionally convergent spectral sequences". In: *Homotopy Invariant Algebraic Structures* 239 (1999), pp. 49–84. DOI: 10.1090/conm/239/03597.

[6] A.K. Bousfield. "The localization of spectra with respect to homology". In: *Topology* 18.4 (1979), pp. 257–281. ISSN: 0040-9383. DOI: https://doi.org/10.1016/0040-9383(79)90018-1. URL: https://www.sciencedirect.com/science/article/pii/0040938379900181.

[7] Daniel Dugger. "Coherence for invertible objects and multigraded homotopy rings". In: *Algebraic & Geometric Topology* 14.2 (Mar. 2014), pp. 1055–1106. DOI: 10.2140/agt.2014.14.1055. URL: https://doi.org/10.2140%2Fagt.2014.14.1055.

[8] Daniel Dugger and Daniel C Isaksen. "Motivic cell structures". In: *Algebraic & Geometric Topology* 5.2 (June 2005), pp. 615–652. DOI: 10.2140/agt.2005.5.615. URL: https://doi.org/10.2140%2Fagt.2005.5.615.

[9] Daniel Dugger and Daniel C Isaksen. "The motivic Adams spectral sequence". In: *Geometry & Topology* 14.2 (2010), pp. 967–1014. DOI: 10.2140/gt.2010.14.967. URL: https://doi.org/10.2140/gt.2010.14.967.

[10] Daniel Dugger et al. *The Multiplicative Structures on Motivic Homotopy Groups*. 2022. arXiv: 2212.03938 [math.AG].

[11] Joseph Hlavinka. *Motivic homotopy theory and cellular schemes*. 2021. URL: http://math.uchicago.edu/~may/REU2021/REUPapers/Hlavinka.pdf.

[12] Mark Hovey. *Model Categories*. American Mathematical Society, 1999.

[13] Daniel C. Isaksen, Guozhen Wang, and Zhouli Xu. *Stable homotopy groups of spheres: From dimension 0 to 90*. 2023. arXiv: 2001.04511 [math.AT].

[14] Marc Levine. "A comparison of motivic and classical stable homotopy theories". In: *Journal of Topology* 7.2 (Aug. 2013), pp. 327–362. DOI: 10.1112/jtopol/jtt031. URL: https://doi.org/10.1112%2Fjtopol%2Fjtt031.

[15] Saunders Mac Lane. *Categories for the Working Mathematician*. New York, NY: Springer New York, 1978. URL: https://doi.org/10.1007/978-1-4757-4721-8.

[16] Saunders MacLane. "Natural Associativity and Commutativity". In: *Rice Institute Pamphlet - Rice University Studies* 4 (1963). URL: https://hdl.handle.net/1911/62865.

[17] Michael A Mandell and J Peter May. *Equivariant orthogonal spectra and S-modules*. American Mathematical Soc., 2002.

[18] J.P. May. "The Additivity of Traces in Triangulated Categories". In: *Advances in Mathematics* 163.1 (2001), pp. 34–73. ISSN: 0001-8708. DOI: https://doi.org/10.1006/aima.2001.1995. URL: https://www.sciencedirect.com/science/article/pii/S0001870801919954.

[19] Amnon Neeman. *Triangulated Categories*. Vol. 148. Princeton Univ. Press, 2001.

[20] nLab authors. *adjoint equivalence*. https://ncatlab.org/nlab/show/adjoint+equivalence. Revision 17. July 2023.

[21] nLab authors. *derived functor in homological algebra*. https://ncatlab.org/nlab/show/derived+functor+in+homological+algebra. Revision 26. Sept. 2023.

[22] nLab authors. *Introduction to Stable Homotopy Theory*. Revision 213. July 2023. URL: https://ncatlab.org/nlab/show/Introduction+to+Stable+Homotopy+Theory.

[23] nLab authors. *Introduction to Stable homotopy theory – 1-1*. https://ncatlab.org/nlab/show/Introduction+to+Stable+homotopy+theory+--+1-1. Revision 42. Oct. 2023.

[24] nLab authors. *Introduction to Stable homotopy theory – 1-2*. https://ncatlab.org/nlab/show/Introduction+to+Stable+homotopy+theory+--+1-2. Revision 77. July 2023.

[25] nLab authors. *Introduction to the Adams Spectral Sequence*. https://ncatlab.org/nlab/show/Introduction+to+the+Adams+Spectral+Sequence. Revision 62. July 2023.

[26] D. C. Ravenel. *Complex cobordism and stable homotopy groups of spheres*. American Mathematical Society, 2004.

[27] Emily Riehl. *Category Theory in Context*. Dover Publications, Inc, 2016.

[28] John Rognes. *Spectral Sequences*. URL: https://www.uio.no/studier/emner/matnat/math/MAT9580/v21/dokumenter/spseq.pdf.

[29] Vladimir Voevodsky. "A1-homotopy theory". In: *Proceedings of the international congress of mathematicians*. Vol. 1. Berlin. 1998, pp. 579–604.

[30] Glen Matthew Wilson and Paul Østvær. "Two-complete stable motivic stems over finite fields". In: *Algebraic & Geometric Topology* 17.2 (2017), pp. 1059–1104. DOI: 10.2140/agt.2017.17.1059. URL: https://doi.org/10.2140/agt.2017.17.1059.