To give a rigorous proof of this claim is rather subtle, so we provide a fully rigorous argument here.

3.19. Associate to each sequence $a = (\alpha_n)$ in which α_n is 0 or 2, the real number

$$x(a) = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n}.$$

Prove the set of all $x(a)$ is precisely the Cantor set described in Theorem 2.44.

Proof. In what follows, let $E_1 \supseteq E_2 \supseteq \cdots$ denote the sets defined in Theorem 2.44, so that the Cantor set may be written as $P = \bigcap_{n\in\mathbb{N}} E_n$, and each E_n is a finite union of 2^n disjoint closed intervals $I_{n,1}, I_{n,2}, \ldots, I_{n,2^n} \subseteq [0, 1]$ of length $1/3^n$:

$$E_n = \bigcup_{k=1}^{2^n} I_{n,k}.$$

We will assume that the $I_{n,k}$'s are indexed in increasing order, so that $\sup I_{n,k} < \inf I_{n,k+1}$ for all $1 \le k < 2^n$. Moreover, note given a sequence $a = (\alpha_n)$ of 0's and 2's, we have that $0 \le \alpha_n/3^n \le 2/3^n$, and we know

$$\sum_{n=1}^{\infty} \frac{2}{3^n} = 2\sum_{n=1}^{\infty} \frac{1}{3^n}$$

converges by Theorem 3.28, so that $x(a) = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n}$ converges by the comparison test (Theorem 3.25).

Now, let $a = (\alpha_n)$ be a sequence of 0's and 2's, then we claim that $x := x(a) \in P$. For each $n \in \mathbb{N}$, let

$$x_n := \sum_{k=1}^{n} \frac{\alpha_k}{3^k},$$

so that $x_n \to x$. Then we claim that for each $n \in \mathbb{N}$, there exists some $1 \le k_n \le 2^n$ such that $x_n = \inf I_{n,k_n}$. We prove this by induction. In the case $n = 1$, we have that $x_1 = 0$ or $x_1 = 2/3$, and recall $E_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$, so the base case is satisfied. Now, suppose that $x_n = \inf I_{n,k_n}$ for some $1 \le k_n \le 2^n$, so that

$$I_{n,k_n} = \left[x_n, x_n + \frac{1}{3^n}\right].$$

Note that by how the E_n's are defined, the intervals obtained by removing the middle one third of I_{n,k_n} are intervals in E_{n+1}, so that there exists $1 \le k_{n+1} \le 2^{n+1}$ such that

$$I_{n+1,k_{n+1}} = \left[x_n, x_n + \frac{1}{3^{n+1}}\right] \qquad \text{and} \qquad I_{n+1,k_{n+1}+1} = \left[x_n + \frac{2}{3^{n+1}}, x_n + \frac{1}{3^n}\right].$$

Finally, note that by how the x_n's are defined,

$$x_{n+1} = x_n + \frac{0}{3^{n+1}} = x_n = \inf I_{n+1,k_{n+1}} \qquad \text{or} \qquad x_{n+1} = x_n + \frac{2}{3^{n+1}} = \inf I_{n+1,k_{n+1}+1}.$$

Thus, we've proven x_{n+1} is the leftmost endpoint of some interval making up E_{n+1}, as desired. In particular, we've shown that $x_n \in E_n$ for all n. Let $N \in \mathbb{N}$. Since $E_N \supseteq E_n$ for all $n \ge N$, we have that $x_n \in E_N$ for all $n \ge N$. Thus since E_N is closed (it is a finite union of closed intervals) and $x_n \to x$, it follows by the definition of a limit that $x \in E_N$. Our choice of $N \in \mathbb{N}$ was arbitrary, and we showed that $x \in E_N$, meaning $x \in \bigcap_{N\in\mathbb{N}} E_N = P$, as desired.

On the other hand, let x be a point in the Cantor set, so that $x \in E_n$ for all $n \in \mathbb{N}$. We inductively define a sequence $a = (\alpha_n)$ satisfying the following properties:

(i) For each $n \in \mathbb{N}$, $\alpha_n \in \{0, 2\}$.
(ii) For each $n \in \mathbb{N}$, there exists some $1 \leq k_n \leq 2^n$ such that $x \in I_{n,k_n}$ and $x_n := \sum_{i=1}^{n} \frac{\alpha_i}{3^i} = \inf I_{n,k_n}$.

To start, note that $x \in E_1$, so either $x \in \left[0, \frac{1}{3}\right]$ or $x \in \left[\frac{2}{3}, 1\right]$. In the former case, set $\alpha_1 = 0$, and in the latter case set $\alpha_1 = 2$. Then clearly (i) and (ii) above are satisfied, as $x_1 = \alpha_1/3$. Now supposing we've defined $\alpha_1, \ldots, \alpha_n$ for some $n \in \mathbb{N}$ so that (i) and (ii) above are satisfied, we defined α_{n+1} as follows. By our induction hypothesis, there exists $1 \leq k_n \leq 2^n$ such that $x \in I_{n,k_n}$ and $x_n = \inf I_{n,k_n}$, so that

$$I_{n,k_n} = \left[x_n, x_n + \frac{1}{3^n}\right].$$

Then since $x \in E_{n+1}$, there exists some $1 \leq k_{n+1} \leq 2^{n+1}$ such that $x \in I_{n+1,k_{n+1}}$. Moreover, since $I_{n,k_n} \cap I_{n+1,k_{n+1}} \neq \emptyset$, by how E_{n+1} is constructed from E_n, we have that either

$$I_{n+1,k_{n+1}} = \left[x_n, x_n + \frac{1}{3^{n+1}}\right] \qquad \text{or} \qquad I_{n+1,k_{n+1}} = \left[x_n + \frac{2}{3^{n+1}}, x_n + \frac{1}{3^n}\right]$$

In the former case, set $\alpha_{n+1} = 0$, so that $x_{n+1} = x_n = \inf I_{n+1,k_{n+1}}$, as desired. In the latter case, set $\alpha_{n+1} = 2$, so that $x_{n+1} = x_n + \frac{2}{3^{n+1}} = \inf I_{n+1,k_{n+1}}$, as desired.

Now that we've constructed the sequence, let $\varepsilon > 0$, and pick N large enough so that $1/3^N < \varepsilon$ (e.g., take $N > \log_3(1/\varepsilon)$). Then given $n \geq N$, by construction we have that x_n and x both live in the interval I_{n,k_n}, which is of length $1/3^n \leq 1/3^N < \varepsilon$, so that in particular for all $n \geq N$ we have $d(x_n, x) < \varepsilon$. Thus

$$x(a) = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n} = \lim_{n \to \infty} x_n = x,$$

as desired. □