3.5. For any two real sequences (a_n), (b_n), prove that

$$\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n,$$

provided the sum on the right is not of the form $\infty - \infty$.

Proof. Write $A := \limsup_{n\to\infty} a_n$, $B := \limsup_{n\to\infty} b_n$, and $C := \limsup_{n\to\infty}(a_n + b_n)$. First, if either A or B equals ∞, then since $\{A, B\} \ne \{\infty, -\infty\}$ we have $A+B = \infty$, so the inequality clearly holds. It remains to consider the case $A, B < \infty$.

Now, consider the case that at least one of A or B equals $-\infty$. By symmetry, we may assume without loss of generality that $A = -\infty$ and $B < \infty$, so that $A + B = -\infty$. Then by Rudin Theorem 3.17, in order to show $C = -\infty$, it suffices to show $a_n + b_n \to -\infty$. To that end, let $M \in \mathbb{R}$. Since $B \in [-\infty, \infty)$, there exists some $x \in \mathbb{R}$ such that $x > B$ (e.g., take $x = \max(0, B + 1)$), so that by Rudin Theorem 3.17 applied to both (a_n) and (b_n), there exists some natural number N such that for all $n \ge N$,

$$a_n < M - x \qquad \text{and} \qquad b_n < x.$$

Then given $n \ge N$, we have

$$a_n + b_n < M - x + x = M,$$

so that $a_n + b_n \to -\infty$, as desired.

Finally, consider the case that that $A, B \in \mathbb{R}$, and let $\varepsilon > 0$. Then by Rudin Theorem 3.17 applied to (a_n) and (b_n), there exists some natural number N such that for all $n \ge N$,

$$a_n < A + \frac{\varepsilon}{2} \qquad \text{and} \qquad b_n < B + \frac{\varepsilon}{2},$$ [1]

so that for all $n \ge N$, we have

$$a_n + b_n < A + B + \varepsilon.$$

Thus by Rudin Theorem 3.19, we have that

$$C = \limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty}(A + B + \varepsilon) = A + B + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $C \le A + B$, as desired. □

[1] Note here we are using that $A, B > -\infty$, as otherwise we wouldn't have $A + \frac{\varepsilon}{2} > A$ and $B + \frac{\varepsilon}{2} > B$, so we couldn't apply Theorem 3.17 (and these inequalities wouldn't even make sense).