# Category Theory

Categories, functors, natural transformations, and (co)limits

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# Category Definition

A *category* $\mathcal{C}$ consists of the following data

- A set $\mathrm{Ob}(\mathcal{C})$, whose elements are called the *objects* of $\mathcal{C}$.
- For any two objects $x, y \in \mathrm{Ob}(\mathcal{C})$, a set $\mathcal{C}(x, y)$ (also denoted $\mathrm{Hom}_{\mathcal{C}}(x, y)$) of *morphisms* from $x$ to $y$. We will often draw a morphism $f \in \mathcal{C}(x, y)$ as an arrow

$$f : x \longrightarrow y$$

  We call $\mathcal{C}(x, y)$ a *hom-set*.
- For every triple of objects $x$, $y$, and $z$ in $\mathrm{Ob}(\mathcal{C})$, a function

$$- \circ - : \mathcal{C}(y, z) \times \mathcal{C}(x, y) \to \mathcal{C}(x, z)$$

  called *composition*.
- For every objects $x$ in $\mathrm{Ob}(\mathcal{C})$, a distinguished morphism $\mathrm{id}_x \in \mathcal{C}(x, x)$ called the *identity morphism* on $x$.

# Category Definition (Cont.)

Together, these data must satisfy:

- The identity morphism is unital w.r.t. composition. In other words, given a morphism $f \in \mathcal{C}(x, y)$, it must be true that

$$f \circ \mathrm{id}_x = f = \mathrm{id}_y \circ f.$$

- Composition must be associative. In other words, given morphisms $f : z \to w$, $g : y \to z$, and $h : x \to y$, it must be true that:

$$f \circ (g \circ h) = (f \circ g) \circ h.$$

# WARNINGS

- A "morphism" is to a hom-set as a vector is to a vector space. They are nothing more than elements of hom-sets!
- Morphisms are *not* functions (at least in general).
- Hom-sets, and by extension morphisms, are *not* unique! It is possible to have two equal hom-sets $\mathcal{C}(x, y) = \mathcal{C}(a, b)$, even if $x \neq a$ or $y \neq b$!

# Category Examples

1. The category Set, whose objects are sets and morphisms are regular set functions.
2. The category CRing, whose objects are commutative rings and morphisms are ring morphisms.
3. The category Ab, whose objects are abelian groups and morphisms are group homomorphisms.
4. Given a commutative ring $R$, we can define the category $R$-Mod, whose objects are $R$-modules and morphisms are $R$-linear maps.
5. The category Top whose objects are topological spaces and morphisms are continuous maps.
6. The category Man whose objects are smooth manifolds and morphisms are smooth maps.
7. The category Ord whose objects are preorders and morphisms are monotone (order-preserving) maps.

# Set-Theoretic Issues

- None of the previous examples are actually categories, by how I've defined it.
- There is no "set of all sets".
- Solution: we have a notion of "small" vs. "big" sets, where the set of all small sets is a big set.
- Set is really the category of all *small* sets. Similarly, Ab is the category whose objects are small abelian groups, etc.
- Definition: We say a category $\mathcal{C}$ is *small* if $\mathrm{Ob}(\mathcal{C})$ is a small set.
- None of the previously listed categories are small.

# Category Examples Cont.

1. Given a category $\mathcal{C}$, we can define a category $\mathcal{C}^{\mathrm{op}}$ by $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) := \mathrm{Ob}(\mathcal{C})$ and $\mathcal{C}^{\mathrm{op}}(x, y) := \mathcal{C}(y, x)$. Then composition in $\mathcal{C}^{\mathrm{op}}$ is simply composition in $\mathcal{C}$.
2. Given a group $(G, \cdot, e)$, we can define a category $BG$ with $\mathrm{Ob}(BG) := \{*\}$, and $BG(*, *) := G$. Composition is defined by $g \circ h := g \cdot h$ (so that $e$ is the identity morphism on $*$).
3. Given a preorder $(P, \leq)$ (so $\leq$ is a transitive and reflexive relation on $P$), we can define a category $CP$ with $\mathrm{Ob}(CP) := P$ and:

$$CP(x, y) := \begin{cases} \{*\} & x \leq y \\ 0 & \text{else.} \end{cases}$$

## Exercise

Show that there is only one possible way to define composition in $CP$, and that it indeed gives an associative, unital composition operation.

# Category Examples Cont.

4. Given two categories $\mathcal{C}$ and $\mathcal{D}$, we can define the *product category* $\mathcal{C} \times \mathcal{D}$ by $\mathrm{Ob}(\mathcal{C} \times \mathcal{D}) := \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{D})$ and

$$(\mathcal{C} \times \mathcal{D})((c, d), (c', d')) := \mathcal{C}(c, c') \times \mathcal{D}(d, d').$$

   Then composition and identities are defined in the obvious way.
5. Let $(X, \mathcal{U})$ be a topological space. Then we can consider the category $\mathrm{Op}(X)$ whose objects are open sets (so $\mathrm{Ob}(\mathrm{Op}(X)) := \mathcal{U}$) and morphisms are the inclusions of sets.

## Exercise

Show that given a topological space $(X, \mathcal{U})$, that $\mathrm{Op}(X)$ is "the same as" the category $C\mathcal{U}$, when viewing $(\mathcal{U}, \subseteq)$ as a poset.

## Definition

Given a category $\mathcal{C}$, we call a morphism $f : x \to y$ an *isomorphism* if there exists a morphism $f^{-1} : y \to x$ such that $f^{-1} \circ f = \mathrm{id}_x$ and $f \circ f^{-1} = \mathrm{id}_y$. Furthermore, we say that $x$ and $y$ are *isomorphic* in this case, and write $x \cong y$.

## Examples

In Set, the isomorphisms are bijections. In Ab, the isomorphisms are precisely isomorphisms of groups. In Top, the isomorphisms are homeomorphisms.

## Exercise

Prove that given a group $G$, the category $BG$ is a *groupoid*, that is, a category in which every morphism is an isomorphism.
Understand the sentence "A group is a groupoid with a single object."

## Exercise

Understand the sentence "A monoid is a category with a single object." This justifies calling a category a "monoidoid."

# Functor Definition

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ from $\mathcal{C}$ to $\mathcal{D}$ consists of the following data:

- A map of sets $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$.
- For every pair of objects $x, y \in \mathcal{C}$, a map of hom-sets:

$$F_{x,y} : \mathcal{C}(x, y) \longrightarrow \mathcal{D}(F(x), F(y))$$

These data must satisfy:

- The maps $F_{x,y}$ must be *functorial*, that is, for any pair of morphisms $g : x \to y$ and $f : y \to z$, we must have:

$$F_{y,z}(f) \circ F_{x,y}(g) = F_{x,z}(f \circ g).$$

  Note: Often, given a morphism $f : x \to y$, we write $F(f)$ to denote the morphism $F_{x,y}(f)$.
- The assignment must be *unital*, that is, given any object $x$ in $\mathcal{C}$, we must have

$$F_{x,x}(\mathrm{id}_x) = \mathrm{id}_{F(x)}.$$

# Functor Examples

- There is a functor $U : \mathrm{Ab} \to \mathsf{Set}$ (the "forgetful functor") which takes an abelian group $(G, +, 0)$ to its underlying set of elements $G$, and takes a group homomorphism to its underlying set-function.
- Given an $R$ module $N$, we can define a functor $- \otimes_R N : R\text{-Mod} \to R\text{-Mod}$ which takes a module $M$ to the module $M \otimes_R N$ and an $R$-linear map $\varphi : M \to M'$ to the $R$-linear map $\varphi \otimes_R 1 : M \otimes N \to M' \otimes N$.
- Given any category $\mathcal{C}$ and some object $c \in \mathrm{Ob}(\mathcal{C})$, we can define the functor (called the "representable presheaf"):

$$\begin{aligned} h_c : \mathcal{C}^{\mathrm{op}} &\to \mathsf{Set} \\ x &\mapsto \mathcal{C}(x, c) \\ (f \in \mathcal{C}^{\mathrm{op}}(x, y)) &\mapsto (- \circ f : \mathcal{C}(y, c) \to \mathcal{C}(x, c)). \end{aligned}$$

- Given any two categories $\mathcal{C}$ and $\mathcal{D}$ and a distinguished object $d \in \mathrm{Ob}(\mathcal{D})$, we can define the *constant functor* $\underline{d} : \mathcal{C} \to \mathcal{D}$ which sends every object in $\mathcal{C}$ to $d$ and every morphism in $\mathcal{C}$ to the identity morphism on $d$.

### Exercise
Show that given a group $G$, any functor $F : BG \to \text{Set}$ is equivalent to a choice of a set $F(*)$ and a $G$-action on that set.

### Exercise
Show that functors preserve isomorphisms.

### Remark
Given any category $\mathcal{C}$, there exists an identity functor $\mathrm{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ which takes each object and each morphism to itself.

### Exercise
Given functors $G : \mathcal{C} \to \mathcal{D}$ and $F : \mathcal{D} \to \mathcal{E}$, define a suitable notion of the composition $F \circ G : \mathcal{C} \to \mathcal{E}$. Show your construction is associative and unital w.r.t. the identity functor.

### Remark
There exists a category Cat, whose objects are (small) categories and whose morphisms are functors.

# Natural Transformation Defintion

Let $F$ and $G$ be functors $\mathcal{C} \to \mathcal{D}$. Then a *natural transformation* $\eta$ from $F$ to $G$, denoted by $\eta : F \Rightarrow G$, is a collection

$$\{\eta_c : F(c) \to G(c)\}_{c \in \mathrm{Ob}(\mathcal{C})}$$

of morphisms in $\mathcal{D}$, such that for every morphism $f : x \to y$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} F(x) & \xrightarrow{F(f)} & F(y) \\ {\scriptstyle \eta_x}\downarrow & & \downarrow{\scriptstyle \eta_y} \\ G(x) & \xrightarrow{G(f)} & G(y) \end{array}$$

In other words, it must be true that $\eta_y \circ F(f) = G(f) \circ \eta_x$. We call this diagram the *naturality square* for $f$.

# Natural Transformation Examples

1. Given a group $G$ and functors $X, Y : BG \rightrightarrows \mathsf{Set}$, a natural transformation $\eta : X \Rightarrow Y$ is the data of an *equivariant map* $\eta_* : X(*) \to Y(*)$.
2. Given any functor $F : \mathcal{C} \to \mathcal{D}$, there is an "identity" natural transformation $\mathrm{id}^F : F \Rightarrow F$ whose component at $x \in \mathrm{Ob}(\mathcal{C})$ is the identity morphism $\mathrm{id}_{F(x)} : F(x) \to F(x)$ in $\mathcal{D}$.
3. Given functors $F, G, H : \mathcal{C} \to \mathcal{D}$ and natural transformations $\mu : F \Rightarrow G$, $\eta : G \Rightarrow H$, we can define their "composition" $\eta \circ \mu : F \Rightarrow H$ whose component at $c \in \mathrm{Ob}(\mathcal{C})$ is given by
$$(\eta \circ \mu)_c := \eta_c \circ \mu_c.$$

### Exercise

Check that the above construction indeed defines a natural transformation, and that this composition operation is associative and unital w.r.t. the identity natural transformation.

### Remark

Given categories $\mathcal{C}$ and $\mathcal{D}$, there exists a category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ (sometimes just denoted $[\mathcal{C}, \mathcal{D}]$ or $\mathcal{D}^{\mathcal{C}}$) whose objects are functors $\mathcal{C} \to \mathcal{D}$ and morphisms are natural transformations between functors.

### Exercise

Show that the isomorphisms in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ are the *natural isomorphisms*, i.e., those natural transformations $\eta$ such that $\eta_c$ is an isomorphism in $\mathcal{D}$ for every $c \in \mathrm{Ob}(\mathcal{C})$.

### Example (The Yoneda Embedding)

Given a category $\mathcal{C}$, the *Yoneda Embedding* is the functor

$$\begin{aligned} \mathcal{Y} : \mathcal{C} &\to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set}) \\ c &\mapsto h_c := \mathcal{C}(-, c). \end{aligned}$$

### Exercise

How can we define the Yoneda embedding on morphisms in $\mathcal{C}$? "Hint": a morphism $f : x \to y$ in $\mathcal{C}$ needs to be sent to a natural transformation $\eta : \mathcal{C}(-, x) \Rightarrow \mathcal{C}(-, y)$.

# Initial & Terminal Objects

## Definition

An *initial object* in a category $\mathcal{C}$ is an object $i \in \mathrm{Ob}(\mathcal{C})$ such that for every object $x \in \mathrm{Ob}(\mathcal{C})$, there exists precisely one morphism $i \to x$.
In other words, $i$ is initial if $\mathcal{C}(i, x)$ is a singleton set for all objects $x$ in $\mathcal{C}$.

## Definition

A *terminal object* in a category $\mathcal{C}$ is an object $t \in \mathrm{Ob}(\mathcal{C})$ such that $\mathcal{C}(x, t)$ is a singleton set for all objects $x$ in $\mathcal{C}$.

## Remark

An object in $\mathcal{C}$ is an initial object if and only if it is a terminal object in $\mathcal{C}^{\mathrm{op}}$, and vice-versa.

# Initial & Terminal Objects Cont.

### Exercise

In a category $\mathcal{C}$, the initial object (resp. terminal object), if it exists, is "unique up to unique isomorphism." In other words, that means if $x$ and $y$ are both initial (resp. terminal) objects in $\mathcal{C}$, then there exists a unique isomorphism $x \to y$.

### Examples

The initial object in Set is the empty set $\emptyset$, while "the" terminal object is the singleton set $\{*\}$, often just denoted $*$.

The initial object and the terminal object in Ab coincide, namely, it is the trivial group.

Given a nontrivial group $G$, the category $BG$ has no initial or terminal object.

### Remark

If a preorder $(P, \leq)$ has a unique maximal element, then that maximal element is the terminal object in $CP$. Similarly, if it has a unique minimal element, then that minimal element is the initial object in $CP$.

# (Co)limits

### Definition

Let $\mathcal{J}$ be a small category and $F : \mathcal{J} \to \mathcal{C}$ be a functor. In the context of (co)limits, we call $F$ a *diagram of shape* $\mathcal{J}$, or just a *diagram*, in $\mathcal{C}$.

### Definition

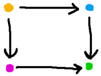
A *cone under* $F$, or a *cocone*, is a pair $(c, \eta)$, where $c \in \mathrm{Ob}(\mathcal{C})$ is an object and $\eta : F \Rightarrow \underline{c}$ is a natural transformation.

### Definition

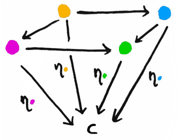
Once again, let $F$ be a diagram of shape $\mathcal{J}$ in $\mathcal{C}$. Then a *cone over* $F$, or just a *cone*, is a pair $(c, \eta)$, where $c \in \mathrm{Ob}(\mathcal{C})$ and $\eta : \underline{c} \Rightarrow F$ is a natural transformation.

# Cone picture

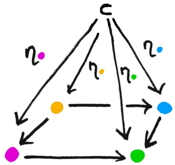
Typically, we imagine $\mathcal{J}$ is a very small category (even finite). Maybe $\mathcal{J}$ looks something like this:



Then if $(c, \eta)$ is a cone under $F$, we have the following image in $\mathcal{C}$.




And if $(c, \eta)$ is a cone over $F$, we have the following image in $\mathcal{C}$.



# (Co)limits Continued

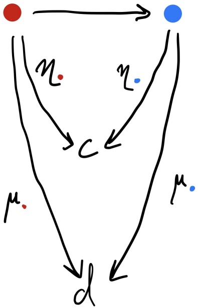
## Definition

Given a diagram $F$ of shape $\mathcal{J}$ in a category $\mathcal{C}$ (so a functor $F : \mathcal{J} \to \mathcal{C}$) and two cocones $(c, \eta)$ and $(d, \mu)$, a *morphism of cocones* is a morphism $f \in \mathcal{C}(c, d)$ such that for all $x \in \mathrm{Ob}(\mathcal{C})$, $\mu_x = f \circ \eta_x$.

# (Co)limits Continued

## Definition

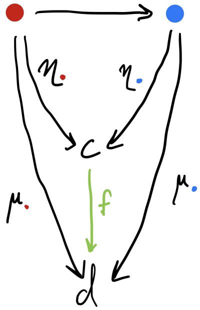
Given a diagram $F$ of shape $\mathcal{J}$ in a category $\mathcal{C}$ (so a functor $F : \mathcal{J} \to \mathcal{C}$) and two cocones $(c, \eta)$ and $(d, \mu)$, a *morphism of cocones* is a morphism $f \in \mathcal{C}(c, d)$ such that for all $x \in \mathrm{Ob}(\mathcal{C})$, $\mu_x = f \circ \eta_x$.



# (Co)limits Continued

## Definition

Given a diagram $F$ of shape $\mathcal{J}$ in a category $\mathcal{C}$ (so a functor $F : \mathcal{J} \to \mathcal{C}$) and two cocones $(c, \eta)$ and $(d, \mu)$, a *morphism of cocones* is a morphism $f \in \mathcal{C}(c, d)$ such that for all $x \in \mathrm{Ob}(\mathcal{C})$, $\mu_x = f \circ \eta_x$.



# (Co)limits Continued

## Definition

Given a diagram $F$ of shape $\mathcal{J}$ in a category $\mathcal{C}$ and two cones $(c, \eta)$, $(d, \mu)$, a *morphism of cones* is a morphism $f \in \mathcal{C}(c, d)$ such that for all $x \in \mathrm{Ob}(\mathcal{C})$, $\mu_x \circ f = \eta_x$.

## Definition

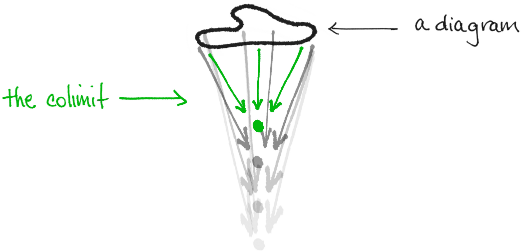
Given a diagram $F$ of shape $\mathcal{J}$ in a category $\mathcal{C}$, define $\mathrm{Cone}_{\mathcal{C}}(F)$ to be the category whose objects are cones under $F$ (cocones) and morphisms are morphisms of cocones, and define $\mathrm{Cone}^{\mathcal{C}}(F)$ to be the category whose objects are cones over $F$ (cones) and morphisms are morphisms of cones.

## Definition

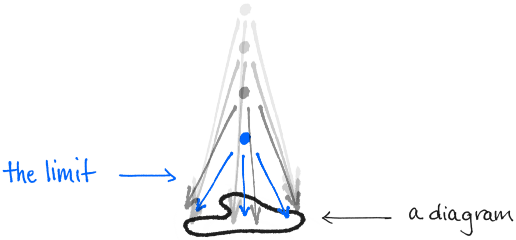
Given a diagram $F$ of shape $\mathcal{J}$ in a category $\mathcal{C}$, the *colimit cone* for $F$ is the initial object in $\mathrm{Cone}_{\mathcal{C}}(F)$ (if it exists), and the *limit cone* for $F$ is the terminal object in $\mathrm{Cone}^{\mathcal{C}}(F)$ (if it exists).

If $(c, \eta)$ is a colimit cone for $F$, then we call the object $c$ the *colimit* of $F$. Similarly, if $(c, \eta)$ is a limit cone for $F$, then we call the object $c$ the *limit* of $F$.

The colimit cone of $F$ is the "shallowest" cone under $F$.




The limit cone of $F$ is the "shallowest" cone over $F$.



# Extra

## Exercise

Given a functor $F : \mathcal{C} \to \mathcal{D}$, how can we define a functor $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$?
Using that definition, given a natural transformation $\eta : F \Rightarrow G$, how can we define a natural transformation $\eta^{\mathrm{op}} : G^{\mathrm{op}} \Rightarrow F^{\mathrm{op}}$?

## Exercise

Let $F$ be a diagram of shape $\mathcal{J}$ in a category $\mathcal{C}$. Then the following are equivalent:

- $(c, \eta)$ is a colimit cone for $F$.
- $(c, \eta^{\mathrm{op}})$ is a limit cone for the functor $F^{\mathrm{op}} : \mathcal{J}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$.

Hence, the limit is the *dual* of the colimit. In other words, the definition of a limit can be obtained by reversing all the arrows in the definition of the colimit.

# Examples of (Co)limits in Set

1. Co(limit) of the empty diagram
2. Co(limit) of a discrete diagram
3. Co(limit) of a (co)span
4. Co(limit) of paralell morphisms
5. Co(limit) of two isomorphic objects

# Direct Limits

## Definition (Directed Set)

A directed $(I, \leq)$ set is a *preorder* (so $\leq$ is a reflexive and transitive relation on $I$) with the additional property that every pair of elements has an upper bound, that is, if $a, b \in I$, then there must exist some $c \in I$ with $a \leq c$ and $b \leq c$.

## Definition (Direct System)

Given a directed set $(I, \leq)$, a *direct system* on $\mathcal{C}$ is a functor $F : CI \to \mathcal{C}$ (a diagram of shape $CI$ in $\mathcal{C}$). We often denote a direct system as a family of objects and morphisms $\langle A_i, f_{ij} \rangle$ indexed by $I$, where the $f_{ij}$'s are viewed like "inclusions."

## Definition (Direct Limit)

Given a direct system $\langle A_i, f_{ij} \rangle$ in a category $\mathcal{C}$ (so the data of a functor $F : CI \to \mathcal{C}$)., the *direct limit* of the system, denoted by

$$\varinjlim A_i,$$

is the **colimit** (!) of the functor $F$.

# Direct Limits Cont.

## Example

An infinite chain of ideals $\mathfrak{p}_1 \subseteq \mathfrak{p}_2 \subseteq \mathfrak{p}_3 \subseteq \cdots$ in a ring $A$ determines a direct system $F : C\mathbb{N} \to A$-Mod (How?).

The direct limit of this system (i.e., the colimit of $F$) is the union $\bigcup_{i=1}^{\infty} \mathfrak{p}_i$.

In a lot of concrete categories, direct limits "act like unions."

# Inverse Limits

## Definition (Inverse Limit)

Given a directed set $(J, \leq)$ and a functor $F : CJ^{\mathrm{op}} \to \mathcal{C}$, the limit of $F$ is often called an *inverse limit*, and is denoted

$$\varprojlim A_j,$$

where the $A_j$ denote the objects in the image of $F$.

Resources:

1. nLab
2. Wikipedia
3. Emily Riehl, *Category Theory in Context*
4. Notes by Walker H. Stern (UVA Postdoc)