THE STANDARD MODEL STRUCTURE ON SPACES

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This document follows Mark Hovey's *Model Categories*, and its intention is to reproduce the proof of the standard model structure on topological spaces in explicit detail. The main result is proven in Theorem 2.3.

1. Preliminaries

We work with von Neumann ordinals, i.e., an ordinal is a transitive set of ordinals (this definition is not circular, the empty set is an ordinal which we call "0"). In the following discussion, let α and β be ordinals. We write $\alpha + 1$ to denote the successor ordinal $\alpha \cup \{\alpha\}$. We write $\alpha < \beta$ to mean $\alpha \in \beta$, and $\alpha \leq \beta$ denotes any of the equivalent conditions: (1) $\alpha < \beta$ or $\alpha = \beta$, (2) $\alpha \in \beta + 1$, (3) $\alpha \subseteq \beta$. Given a collection of ordinals B, we write $\sup B$ or $\sup_{\beta \in B} \beta$ to denote the ordinal $\bigcup_{\beta \in B} \beta$. We define the sum of ordinals α and β recursively: $\alpha + 0 := \alpha$, $\alpha + (\beta + 1) := (\alpha + \beta) + 1$, and $\alpha + \beta := \sup_{\delta < \beta}(\alpha + \delta)$ when β is a limit ordinal. Note that addition of ordinals is not commutative, but it is associative, and continuous in its right argument: given an ordinal α and a collection of ordinals B, $\alpha + \sup B = \sup_{\beta \in B}(\alpha + \beta)$. We say an ordinal λ is a *limit ordinal* if either of the following equivalent conditions hold: (1) $\lambda = \sup_{\beta < \lambda} \beta$ or (2) $\lambda \neq \beta + 1$ for all ordinals β. Note that 0 is a limit ordinal under our definition. We may regard an ordinal α as a poset category, in which case the colimit in α is given by the supremum. We let **Ord** denote the poset category of all (small) ordinals, so there exists a unique arrow $\alpha \to \beta$ if $\alpha \leq \beta$. Given a set X, we write $|X|$ to denote its *cardinality*, i.e., $|X|$ is the least ordinal α such that there exists a bijection between α and X. A cardinal number is an an ordinal which is the cardinality of some set X.

Definition 1.1 (Hovey Definition 2.1.1)**.** Suppose $\mathcal{C}$ is a cocomplete category, and λ is an ordinal. A λ-*sequence* in $\mathcal{C}$ is a colimit-preserving functor $X : \lambda \to \mathcal{C}$, commonly written as

$$X_0 \to X_1 \to \cdots \to X_\beta \to \cdots .$$

Since X preserves colimits, for all limit ordinals $\gamma < \lambda$, the arrows $X_\alpha \to X_\gamma$ for $\alpha < \gamma$ form a colimit cone under $\{X_\alpha\}_{\alpha < \gamma}$. We refer to the map $X_0 \to \operatorname{colim}_{\beta < \lambda} X_\beta$ as the *composition* of the λ-sequence. Given a collection $\mathcal{D}$ of morphisms in $\mathcal{C}$ such that every map $X_\beta \to X_{\beta+1}$ for $\beta + 1 < \lambda$ is in $\mathcal{D}$, we refer to the composition $X_0 \to \operatorname{colim}_{\beta < \lambda} X_\beta$ as a *transfinite composition* of arrows in $\mathcal{D}$.[1]

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[1] To be more precise, there may be different (isomorphic) choices of colimit $\operatorname{colim}_{\beta < \gamma} X_\beta$, which give rise to different choices of composition $X_0 \to \operatorname{colim}_{\beta < \gamma} X_\beta$. Thus, the composition of a λ-sequence is only unique up to composition by a unique isomorphism.

Of particular importance to us will be collections of arrows which are *closed under transfinite composition*, i.e., collections $\mathcal{D}$ for which given any ordinal λ and λ-sequence X of arrows in $\mathcal{D}$, for any choice of colimit $\operatorname{colim} X$, the canonical map $X_0 \to \operatorname{colim} X$ is also in $\mathcal{D}$. We prove the following useful result about when a class of morphisms is closed under transfinite composition:

Lemma 1.2. *Let $\mathcal{C}$ be a category, and $\mathcal{D}$ a collection of arrows in $\mathcal{C}$ satisfying the following properties: $\mathcal{D}$ is closed under composition with isomorphisms, and given an ordinal λ and a λ-sequence $X : \lambda \to \mathcal{C}$ of arrows in $\mathcal{D}$ (so $X_\beta \to X_{\beta+1}$ belongs to $\mathcal{D}$ for all $\beta+1 < \lambda$), if we then get then get for free that $X_\alpha \to X_\beta$ belongs to $\mathcal{D}$ for all $\alpha \leq \beta < \lambda$, then $\mathcal{D}$ is closed under transfinite composition.*

Proof. Let λ be an ordinal, and $X : \lambda \to \mathcal{C}$ a λ-sequence of arrows in $\mathcal{D}$. First, suppose $\lambda = \mu + 1$ is a successor ordinal. Since we know that any transfinite composition of X may be obtained from another by composing with an isomorphism and $\mathcal{D}$ is closed under composition with isomorphisms, it suffices to show there exists *some* transfinite composition of X belonging to $\mathcal{D}$. We know $\sup_{\beta<\lambda} \beta = \sup_{\beta<\mu+1} \beta = \mu$, and X is colimit preserving, so that X_μ is a colimit of the diagram X via the arrows $X_\alpha \to X_\mu$ for $\alpha < \lambda = \mu + 1$. But we know in particular that $X_0 \to X_\mu$ belongs to $\mathcal{D}$, so we are done.

Conversely, suppose λ is a limit ordinal. Let $j : X \Rightarrow \underline{X_\lambda}$ be a colimit cone for X. We may use j to extend X to a $(\lambda+1)$-sequence in the obvious way (so for $\alpha < \lambda$, the structure map $X_\alpha \to X_\lambda$ is given by j and the arrow $X_\lambda \to X_\lambda$ is the identity, as is necessary). Further note that X is still a sequence of arrows in $\mathcal{D}$, as given $\beta + 1 < \lambda + 1$, so $\beta + 1 \leq \lambda$, it is not possible that $\beta + 1 = \lambda$ as λ is a limit ordinal, in which case we know the map $X_\beta \to X_{\beta+1}$ belongs to $\mathcal{D}$ as $\beta + 1 < \lambda$. Hence, unravelling definitions and applying the asserted property of $\mathcal{D}$, we get for free that $j_0 : X_0 \to X_\lambda$ belongs to $\mathcal{D}$. □

Lemma 1.3. *Given a cocomplete category $\mathcal{C}$ and a collection $\mathcal{D}$ of arrows in $\mathcal{C}$, if $\mathcal{D}$ is closed under transfinite composition, then given any limit ordinal λ and λ-sequence $X : \lambda \to \mathcal{C}$, for all $\alpha < \lambda$ the canonical map $X_\alpha \to \operatorname{colim} X$ belongs to $\mathcal{D}$.*

Proof Sketch. Let $\alpha < \lambda$, and fix a colimit cone $j : X \Rightarrow \underline{\operatorname{colim} X}$. Define $S := \{\beta : \alpha \leq \beta \leq \lambda\} \subseteq \lambda + 1$. Define a map $\phi : S \to \mathbf{Ord}$ via transfinite recursion. Let $\phi(\alpha) = 0$. Supposing $\phi(\beta)$ has been defined, let $\phi(\beta+1) = \phi(\beta) + 1$. Finally, supposing $\alpha < \gamma \leq \lambda$ is a limit ordinal and $\phi(\beta)$ has been defined for $\alpha \leq \beta < \gamma$, define $\phi(\gamma) = \sup_{\alpha \leq \beta < \gamma} \phi(\beta)$. It is straightforward to verify that ϕ is order preserving, sends limit ordinals to limit ordinals, and satisfies $\alpha + \phi(\beta) = \beta$ for all $\alpha \leq \beta \leq \lambda$.

Now, construct a $\phi(\lambda)$-sequence $Y : \phi(\lambda) \to \mathcal{C}$ by $Y_\beta := X_{\alpha+\beta}$, and given $\beta \leq \beta' < \phi(\lambda)$, define the map $Y_\beta \to Y_{\beta'}$ to be the arrow $X_{\alpha+\beta} \to X_{\alpha+\beta'}$ for X. Checking that Y is functorial and colimit-preserving follows directly from the fact that X is functorial and colimit-preserving. Then it can be seen that the $j_{\alpha+\beta}$'s for $0 \leq \beta < \phi(\lambda)$ restrict to a colimit cone under Y. Since Y is a $\phi(\lambda)$-sequence in $\mathcal{D}$ and $\mathcal{D}$ is closed under transfinite compositions, it follows that $j_\alpha \in \mathcal{D}$, as desired. □

Definition 1.4 (Hovey Definition 2.1.2)**.** Let γ be a cardinal. An ordinal α is *γ-filtered* if it is a limit ordinal and, if $A \subseteq \alpha$ and $|A| \leq \gamma$, then $\sup A < \alpha$.

Given a cardinal γ, a γ-filtered category $\mathcal{C}$ is one such that any diagram $\mathcal{D} \to \mathcal{C}$ has a cocone when $\mathcal{D}$ has $< \gamma$ arrows. A catgory is just "filtered" if it is ω-filtered, i.e., if every finite diagram in $\mathcal{C}$ admits a cocone. Note that an ordinal α is γ-filtered precisely when it is γ-filtered as a category, and in particular every ordinal is ω-filtered.

Definition 1.5 (Hovey Definition 2.1.3)**.** Suppose $\mathcal{C}$ is a comcomplete category, $\mathcal{D} \subseteq \operatorname{Mor} \mathcal{C}$ is some collection of morphisms of $\mathcal{C}$, A is an object of $\mathcal{C}$, and κ is a cardinal. We say that A is

κ-small relative to $\mathcal{D}$ if, for all κ-filtered ordinals λ and all λ-sequences

$$X_0 \to X_1 \to \cdots \to X_\beta \to \cdots$$

such that each map $X_\beta \to X_{\beta+1}$ is in $\mathcal{D}$ for $\beta + 1 < \lambda$, the canonical map of sets

$$\operatorname*{colim}_{\beta<\lambda} \mathcal{C}(A, X_\beta) \to \mathcal{C}(A, \operatorname*{colim}_{\beta<\lambda} X_\beta)$$

is an isomorphism. We say that A is *small relative to* $\mathcal{D}$ if it is κ-small relative to $\mathcal{D}$ for some κ. We say that A is *small* if it is small relative to $\mathcal{C}$ itself.

Definition 1.6 (Hovey Definition 2.1.4)**.** Suppose $\mathcal{C}$ is a cocomplete category, $\mathcal{D}$ is a collection of morphisms of $\mathcal{C}$, and A is an object of $\mathcal{C}$. We say that A is *finite relative to* $\mathcal{D}$ if A is κ-small relative to $\mathcal{D}$ for some finite cardinal κ. We say A is *finite* if it is finite relative to $\mathcal{C}$ itself. In particular, since *every* limit ordinal is κ-filtered for any finite cardinal κ, for an object A to be finite relative to $\mathcal{D}$, maps from A must commute with colimits of *arbitrary* λ-sequences for every limit ordinal λ.

Remark 1.7. Recall that given a small category $\mathcal{D}$ and a functor $F : \mathcal{D} \to \mathbf{Set}$, we may explicitly construct the colimit of F as the set

$$\operatorname{colim} F := \left(\coprod_{d \in \mathcal{D}} F(d) \right) / \sim,$$

where the equivalence relation $\sim$ is **generated** by

$$((x \in F(d)) \sim (x' \in F(d'))) \quad \text{if} \quad (\exists (f : d \to d') \text{ with } Ff(x) = x').$$

In particular, if $\mathcal{D}$ is a filtered category then the resulting relation can be described as follows:

$$((x \in F(d)) \sim (x' \in F(d'))) \quad \text{iff} \quad (\exists\, d'', (f : d \to d''), (g : d' \to d'') \text{ with } Ff(x) = Fg(x')).$$

Then the colimit cone $\eta : F \Rightarrow \underline{\operatorname{colim} F}$ is defined by $\eta_d(x) = [x]$ for $d \in \mathcal{D}$ and $x \in F(d)$, where $[x]$ denotes the equivalence class of x in $\operatorname{colim} F$. Given a cone $\varepsilon : F \Rightarrow \underline{Y}$ under F, the unique map $\operatorname{colim} F \to Y$ maps an equivalence class $[x]$ represented by an element $x \in F(d)$ to the element $\varepsilon_d(x)$.

Similarly, we may explicitly construct the limit of a functor $F : \mathcal{D} \to \mathbf{Set}$ as the subset

$$\lim F = \left\{ (x_d)_{d \in \mathcal{D}} \in \prod_{d \in \mathcal{D}} F(d) : \forall (d_i \xrightarrow{\alpha} d_j) \in \mathcal{D},\ F(\alpha)(x_{d_i}) = x_{d_j} \right\},$$

in which case the limit cone is simply the restriction of the projection maps for $\prod_{d \in \mathcal{D}} F(d)$ to $\lim F$.

Now we unravel what the "canonical map" of Definition 1.5 is. Suppose we are given a cocomplete category $\mathcal{C}$, an element $A \in \mathcal{C}$, an ordinal λ, and a λ-sequence $X : \lambda \to \mathcal{C}$. For $\alpha \leq \beta < \lambda$, let $\iota_{\alpha,\beta}$ be the map $X_\alpha \to X_\beta$. Let $\eta : X \Rightarrow \underline{\operatorname{colim} X}$ be the colimit cone. By whiskering the colimit cone along the functor $\mathcal{C}(A, -)$, we get a cone $\mathcal{C}(A, \eta) : \{\mathcal{C}(A, X_\beta)\}_{\beta<\lambda} \Rightarrow \underline{\mathcal{C}(A, \operatorname{colim} X)}$. Then if we let $\varepsilon : \{\mathcal{C}(A, X_\beta)\}_{\beta<\lambda} \Rightarrow \underline{\operatorname{colim}_{\beta<\lambda} \mathcal{C}(A, X_\beta)}$ be the colimit cone, the universal property of the colimit gives us the canonical map $\ell : \operatorname{colim}_{\beta<\lambda} \mathcal{C}(A, X_\beta) \to \mathcal{C}(A, \operatorname{colim} X)$, so that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{C}(A,X_0) \arrow[r, "(\iota_{0,1})_*"] \arrow[drr, "\varepsilon_0"] \arrow[ddrr, "(\eta_0)_*"'] & \mathcal{C}(A,X_1) \arrow[r, "(\iota_{1,2})_*"] \arrow[dr, "\varepsilon_1"] \arrow[ddr, "(\eta_1)_*"'] & \cdots \arrow[r] & \mathcal{C}(A,X_\beta) \arrow[r, "(\iota_{\beta,\beta+1})_*"] \arrow[dl, "\varepsilon_\beta"'] \arrow[ddl, "(\eta_\beta)_*"] & \cdots \\
& & \operatorname{colim}_{\beta<\lambda}\mathcal{C}(A,X_\beta) \arrow[d, dashed, "\ell"] & & \\
& & \mathcal{C}(A,\operatorname{colim}X) & &
\end{tikzcd}$$

In particular, by Remark 1.7, we know elements of $\operatorname{colim}_{\beta<\lambda}\mathcal{C}(A,X_\beta)$ are equivalence classes of arrows $f : A \to X_\beta$ for $\beta < \lambda$ under the relation $[f : A \to X_\beta] = [g : A \to X_{\beta'}]$ iff there exists $\beta'' \geq \beta, \beta'$ with $\iota_{\beta,\beta''} \circ f = \iota_{\beta',\beta''} \circ g$, and the map ε_β sends an arrow $f \in \mathcal{C}(A,X_\beta)$ to the element $[f]$. Then it follows that $\ell([f : A \to X_\beta]) = \eta_\beta \circ f$. Thus, this gives us the following result:

Proposition 1.8. *Given a cocomplete category $\mathcal{C}$, a collection $\mathcal{D}$ of arrows in $\mathcal{C}$, an object A in $\mathcal{C}$, and a cardinal κ, A is κ-small relative to $\mathcal{D}$, if, for all κ-filtered ordinals λ and all λ-sequences $X : \lambda \to \mathcal{C}$ such that the map $X_\beta \to X_{\beta+1}$ belongs to $\mathcal{D}$ for all $\beta + 1 < \lambda$, given any colimit $\operatorname{colim} X$ for X, the following holds:*

(i) Given arrows $f : A \to X_\alpha$ and $g : A \to X_\beta$ in $\mathcal{C}$, if f and g agree in the colimit (i.e., if the compositions $A \xrightarrow{f} X_\alpha \to \operatorname{colim} X$ and $A \xrightarrow{g} X_\beta \to \operatorname{colim} X$ are equal), then f and g are equal in some stage of the colimit (i.e., there exists $\gamma < \lambda$ with $\alpha, \beta \leq \gamma$ such that the compositions $A \xrightarrow{f} X_\alpha \to X_\gamma$ and $A \xrightarrow{g} X_\beta \to X_\gamma$ are equal).

(ii) Any arrow $f : A \to \operatorname{colim} X$ factors through some stage of the colimit (i.e., there exists $\beta < \lambda$ and an arrow $\widetilde{f} : A \to X_\beta$ such that the composition $A \xrightarrow{\widetilde{f}} X_\beta \to \operatorname{colim} X$ equals f).

In terms of the canonical map $\operatorname{colim}_{\beta<\lambda}\mathcal{C}(A,X_\beta) \to \mathcal{C}(A,\operatorname{colim} X)$, the first condition shows injectivity, while the second shows surjectivity.

We will use the characterization of smallness given by this remark whenever proving smallness arguments, as in the following example.

Example 1.9 (Hovey 2.1.5)**.** Every set is small. Indeed, if A is a set we claim that A is $|A|$-small. To see this, suppose λ is an $|A|$-filtered ordinal, and X is a λ-sequence of sets. First of all, by Remark 1.7, the elements of $\operatorname{colim} X$ are equivalence classes of elements $a \in X_\alpha$ where $a \in X_\alpha$ and $b \in X_\beta$ represent the same element of $\operatorname{colim} X$ iff there exists $\alpha, \beta \leq \gamma < \lambda$ so that a and b are sent to the same elements by the maps $X_\alpha \to X_\gamma$ and $X_\beta \to X_\gamma$, respectively. Now, we show the conditions of Proposition 1.8.

First, we need to show that given $\alpha, \beta < \lambda$, if $f : A \to X_\alpha$ and $g : A \to X_\beta$ such that the compositions $\overline{f} : A \xrightarrow{f} X_\alpha \to \operatorname{colim} X$ and $\overline{g} : A \xrightarrow{g} X_\beta \to \operatorname{colim} X$ are equal, then f and g are equal in some stage of the colimit. For each $a \in A$, since $\overline{f}(a) = \overline{f}(g)$ in $\operatorname{colim} X$, by the above characterization of $\operatorname{colim} X$, there exists $\gamma_a < \lambda$ with $\alpha, \beta \leq \gamma_a$ such that $f(a)$ and $g(a)$ are sent to the same element in X_{γ_a} by the maps $X_\alpha \to X_{\gamma_a}$ and $X_\beta \to X_{\gamma_a}$, respectively. Then let $\gamma := \sup_{a\in A} \gamma_a$. Since $|\{\gamma_a\}_{a\in A}| \leq |A|$ and λ is $|A|$-filtered, necessarily $\gamma < \lambda$. Then clearly the compositions $A \xrightarrow{f} X_\alpha \to X_\gamma$ and $A \xrightarrow{g} X_\beta \to X_\gamma$ agree for all $a \in A$.

Secondly, we wish to show that given a map $f : A \to \operatorname{colim} X$, that f factors through $X_\beta \to \operatorname{colim} X$ for some $\beta < \lambda$. For each $a \in A$, by the explicit description of $\operatorname{colim} X$, there exists some

$\beta_a < \lambda$ and some $x_a \in X_{\beta_a}$ such that $f(a) = [x_a]$. Then let $\beta := \sup_{a \in A} \beta_a$, so $\beta < \lambda$ as X is $|A|$-filtered. Now define $\widetilde{f} : A \to X_\beta$ like so: for $a \in A$, define $\widetilde{f}(a) \in X_\beta$ to be the image of x_a along the map $X_{\beta_a} \to X_\beta$. Then clearly the composition $f' : A \xrightarrow{\widetilde{f}} X_\beta \to \operatorname{colim} X$ is equal to f, by unravelling definitions.

Definition 1.10 (Hovey Definition 2.1.7)**.** Let I be a class of maps in a category $\mathcal{C}$.

(1) A map is *I-injective* if it has the right lifting property w.r.t. every map in I. The class of I-injective maps is denoted I-inj (or $I_\perp$).
(2) A map is *I-projective* if it has the left lifting property w.r.t. every map in I. The class of I-projective maps is denoted I-proj (or ${}_\perp I$).
(3) A map is an *I-cofibration* if it has the left lifting property w.r.t. every I-injective map. The class of I-cofibrations is the class (I-inj)-proj and is denoted I-cof (or ${}_\perp(I_\perp)$).
(4) A map is an *I-fibration* if it has the right lifting property w.r.t. every I-projective map. The class of I-fibrations is the class (I-proj)-inj and is denoted I-fib (or $({}_\perp I)_\perp$).

The following is asserted in Hovey on pg. 30 following Definition 2.1.7, but not proven. We provide a proof.

Lemma 1.11. *Given classes A and B of maps in a category $\mathcal{C}$ with $A \subseteq B$, we have $A \subseteq {}_\perp(A_\perp)$, $A \subseteq ({}_\perp A)_\perp$, $({}_\perp(A_\perp))_\perp = A_\perp$, ${}_\perp(({}_\perp A)_\perp) = {}_\perp A$, $A_\perp \supseteq B_\perp$, ${}_\perp A \supseteq {}_\perp B$, ${}_\perp(A_\perp) \subseteq {}_\perp(B_\perp)$, and $({}_\perp A)_\perp \subseteq ({}_\perp B)_\perp$.*

Proof. Each of these amount to unravelling definitions and are entirely straightforward. □

Definition 1.12 (Hovey Definition 2.1.9)**.** Let I be a set of maps in a cocomplete category $\mathcal{C}$. A *relative I-cell complex* is a transfinite composition of pushouts of elements of I. That is, if $f : A \to B$ is a relative I-cell complex, then there is an ordinal λ and a λ-sequence $X : \lambda \to \mathcal{C}$ such that f is the composition of X and such that, for each β such that $\beta + 1 < \lambda$, there is a pushout square

$$\begin{tikzcd}
C_\beta \arrow[r] \arrow[d, "g_\beta"'] & X_\beta \arrow[d] \\
D_\beta \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & X_{\beta+1}
\end{tikzcd}$$

with $g_\beta \in I$. We denote the collection of relative I-cell complexes by I-cell. We say that $A \in \mathcal{C}$ is an *I-cell complex* if the map $0 \to A$ is a relative I-cell complex.

Lemma 1.13. *Let $\mathcal{C}$ be a category and I a class of morphisms in $\mathcal{C}$. Then* I-cell *is closed under composition with isomorphisms.*

Proof Sketch. Suppose that $f : B \to C$ is an element of I-cell, and $h : A \to B$ and $g : C \to D$ are isomorphisms in $\mathcal{C}$. We wish to show $f \circ h$ and $g \circ f$ are also elements of I-cell. Since $f \in I$-cell, there exists an ordinal λ, a λ-sequence X with $X_0 = B$, and a colimit cone $\eta : X \Rightarrow \underline{C}$, such that $\eta_0 = f$.

First of all, construct a new cone $\eta' : X \Rightarrow \underline{D}$ under X where $\eta'_\beta := g \circ \eta_\beta$. It is straightforward to verify that η' is a colimit cone for X since η is a colimit cone and g is an isomorphism. Thus, $g \circ f = g \circ \eta_0 = \eta'_0 \in I$-cell, as η'_0 is the composition of a sequence of pushouts of elements of I.

On the other hand, we may construct a new λ-sequence X' by defining $X'_0 = A$, $X'_\beta = X_\beta$ for all $0 < \beta < \lambda$, the map $X'_0 \to X'_\beta$ for $0 < \beta < \lambda$ to be the composition

$$A \xrightarrow{h} B = X_0 \longrightarrow X_\beta,$$

and the composition $X'_\alpha \to X'_\beta$ to simply be the same map $X_\alpha \to X_\beta$ for $0 < \alpha \leq \beta < \lambda$. It is straightforward to verify that defines a λ-sequence, and that we may define a colimit cone

$\eta' : X' \Rightarrow \underline{C}$ by $\eta'_0 = \eta_0 \circ h = f \circ h$, and $\eta'_\beta = \eta_\beta$ for $0 < \beta < \lambda$. Furthermore, clearly for all $1 < \beta + 1 < \lambda$, we have the arrow $X'_\beta \to X'_{\beta+1}$ is a pushout of a map in I. Thus, in order to show $f \circ h \in I$-cell, it remains to show that the arrow $X'_0 = A \to X_1 = X'_1$ is a pushout of a map in I. Indeed, we know $B = X_0 \to X_1$ is a pushout of a map $k : P \to Q$ in I, and it can be easily verified the diagram on the right is a pushout diagram as the left diagram is a pushout diagram and h is an isomorphism

$$\begin{tikzcd} P \arrow[r] \arrow[d, "k"'] & X_0 \arrow[d] \\ Q \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & X_1 \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} P \arrow[r] \arrow[dd] & X_0 \arrow[r, "h^{-1}"] & X'_0 \arrow[d, "h"] \\ & & X_0 \arrow[d] \\ Q \arrow[rr] & & X'_1 \arrow[ul, phantom, "\lrcorner", very near end] \end{tikzcd}$$

□

Definition 1.14. Let $\mathcal{C}$ be a category and I a collection of morphisms in $\mathcal{C}$. Then if I is closed under transfinite composition, pushouts, and retracts then we say I is *saturated.*

Lemma 1.15. *Suppose I is a class of maps in a cocomplete category $\mathcal{C}$. Then ${}_\perp I$ is saturated.*

TODO

Proof. □

This yields the following Corollary:

Corollary 1.16 (Hovey 2.1.10)**.** *Given a cocomplete category $\mathcal{C}$ and a class of maps I in $\mathcal{C}$, I-cell $\subseteq {}_\perp(I_\perp)$.*

Theorem 1.17 (Small Object Argument, Hovey 2.1.14)**.** *Suppose $\mathcal{C}$ is a cocomplete categroy, and I is a set of maps in $\mathcal{C}$. Suppose the domains of the maps of I are small relative to I-cell. Then there is a functorial factorization (γ, δ) on $\mathcal{C}$ such that for all morphisms $f \in \mathcal{C}$, the map $\gamma(f)$ is in I-cell and the map $\delta(f)$ is in I-inj.*

TODO

Proof. □

Corollary 1.18 (Hovey 2.1.15)**.** *Suppose that I is a set of maps in a cocomplete category $\mathcal{C}$. Suppose as well that the domains of I are small relative to I-cell. Then given $f : A \to B$ in ${}_\perp(I_\perp)$, there is a $g : A \to C$ in I-cell such that f is a retract of g by a map which fixes A.*

TODO

Proof. □

Definition 1.19 (Hovey Definition 2.1.17)**.** Suppose $\mathcal{C}$ is a model category. We say that $\mathcal{C}$ is *cofibrantly generated* if there are sets I and J of maps such that:

1. The domains of the maps of I are small relative to I-cell;
2. The domains of the maps of J are small relative to J-cell;
3. The class of fibrations is $J_\perp$; and
4. The class of trivial fibrations is $I_\perp$.

We refer to I as the set of *generating cofibrations* and to J as the set of *generating trivial cofibrations.* A cofibrantly generated model category is *finitely generated* if we can choose the sets I and J above so that the domains and codomains of I and J are finite relative to I-cell.

Proposition 1.20 (Hovey Proposition 2.1.18)**.** *Suppose $\mathcal{C}$ is a cofibrantly generated model category, with generating cofibrations I and generating trivial fibrations J.*

- *(a) The cofibrations form the class ${}_\perp(I_\perp)$.*
- *(b) Every cofibration is a retract of a relative I-cell complex.*
- *(c) The domains of I are small relative to the cofibrations.*
- *(d) The trivial cofibrations form the class ${}_\perp(J_\perp)$.*
- *(e) Every trivial cofibration is a retract of a relative J-cell complex.*

(f) The domains of J are small relative to the trivial cofibrations.

If $\mathcal{C}$ is fibrantly generated, then the domains and codomains of I and J are finite relative to the cofibrations.

Proof. ☐ TODO

Theorem 1.21 (Hovey Theorem 2.1.19)**.** *Suppose $\mathcal{C}$ is a complete & cocomplete category. Suppose $\mathcal{W}$ is a subcategory of $\mathcal{C}$, and I and J are sets of maps of $\mathcal{C}$. Then there is a cofibrantly generated model structure on $\mathcal{C}$ with I as the set of generating cofibrations, J as the set of generating trivial fibrations, and $\mathcal{W}$ as the subcategory of weak equivalences if and only if the following conditions are satisfied.*

1. *The subcategory $\mathcal{W}$ has the 2-of-3 property and is closed under retracts.*
2. *The domains of I are small relative to I-cell.*
3. *The domains of J are small relative to J-cell.*
4. *J-cell $\subseteq \mathcal{W} \cap {}_{\perp}(I_{\perp})$.*
5. *$I_{\perp} \subseteq \mathcal{W} \cap J_{\perp}$.*
6. *Either $\mathcal{W} \cap {}_{\perp}(I_{\perp}) \subseteq {}_{\perp}(J_{\perp})$ or $\mathcal{W} \cap J_{\perp} \subseteq I_{\perp}$.*

Proof. ☐ TODO

We establish some notation for the following results. Given an adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$, we will use "$(-)^{\sharp}$" and "$(-)^{\flat}$" to decorate a pair of adjoint arrows $f^{\sharp} : F(C) \to D$ and $f^{\flat} : C \to G(D)$. Ocasionally we will write $g^{\sharp}$ or $g^{\flat}$ to denote the transpose of a morphism $g : C \to G(D)$ or $g : F(C) \to D$ not already written in this form.

Given a complete and cocomplete category $\mathcal{C}$ and arrows $i : A \to B$, $j : C \to D$, $p : X \to Y$, where C and D are exponentiable,[2] define $Q(j,p)$ to be the fiber product (j^*, p_*) which fits into the following fiber diagram:

$$\begin{tikzcd}
X^D \arrow[dr, dashed, "Q(j,p)"] \arrow[drr, bend left, "p_*"] \arrow[ddr, bend right, "j^*"'] & & \\
& X^C \times_{Y^C} Y^D \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y^D \arrow[d, "j^*"] \\
& X^C \arrow[r, "p_*"] & Y^C
\end{tikzcd}$$

where given an object Z, the pullback map $j^* : Z^D \to Z^C$ is obtained as the adjoint of the composition

$$Z^D \times C \xrightarrow{\mathrm{id} \times j} Z^D \times D \xrightarrow{\varepsilon_Z} Z,$$

where ε is the counit of the adjunction $- \times D \dashv (-)^D$.

Similarly, write $i \wedge j := (i \times \mathrm{id}_D, \mathrm{id}_B \times j)$ to be the arrow which fits into the following pushout diagram:

$$\begin{tikzcd}
A \times C \arrow[r, "i \times \mathrm{id}_C"] \arrow[d, "\mathrm{id}_A \times j"'] & B \times C \arrow[d] \arrow[ddr, bend left, "\mathrm{id}_B \times j"] & \\
A \times D \arrow[r] \arrow[drr, bend right, "i \times \mathrm{id}_D"'] & A \times D \coprod_{A \times C} B \times C \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "i \wedge j"] & \\
& & B \times D
\end{tikzcd}$$

---

[2] Explicitly, the functors $- \times C$ and $- \times D$ admit right adjoints $(-)^C$ and $(-)^D$, respectively.

Proposition 1.22. *Let $\mathcal{C}$ be a complete and cocomplete category, and suppose we are given arrows $i : A \to B$, $j : C \to D$, and $p : X \to Y$ with C and D exponentiable objects in $\mathcal{C}$. Then there is a bijective correspondence between lifting problems of the form*

(1)
$$\begin{tikzcd} A \times D \coprod_{A\times C} B \times C \arrow[r] \arrow[d, "i\wedge j"'] & X \arrow[d, "p"] \\ B \times D \arrow[r] & Y \end{tikzcd}$$

and lifting problems of the form

(2)
$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X^D \arrow[d, "Q(j,p)"] \\ B \arrow[r] & X^C \times_{Y^C} Y^D. \end{tikzcd}$$

Moreover, this bijection extends to a bijection between the solutions of (1) and the solutions of (2).

Before we prove this proposition, we first recall two results

Lemma 1.23. *Given a pair of adjoint functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ (F is the left adjoint), for any morphisms with domains and codomains as displayed below*

$$\begin{tikzcd} F(C) \arrow[r, "f^\sharp"] \arrow[d, "Fh"'] & D \arrow[d, "k"] & & C \arrow[r, "f^\flat"] \arrow[d, "h"'] & G(D) \arrow[d, "Gk"] \\ F(C') \arrow[r, "g^\sharp"'] & D' \arrow[r, phantom, "\leftrightsquigarrow"] & {} & C' \arrow[r, "g^\flat"'] & G(D') \end{tikzcd}$$

the left-hand square commutes in $\mathcal{D}$ iff the right-hand transposed square commutes in $\mathcal{C}$.

Add reference

Proof. This is Lemma 4.1.3 from Riehl. $\square$

Lemma 1.24. *Let $\mathcal{C}$ be a complete and cocomplete category, and suppose we are given morphisms $j : C \to D$ and $k^\sharp : Z \times D \to W$ in $\mathcal{C}$ with C and D exponentiable. Then the following two diagrams commute*

$$\begin{tikzcd} Z \times C \arrow[r, "\mathrm{id}\times j"] \arrow[d, "k^\flat\times\mathrm{id}"'] & Z \times D \arrow[d, "k^\sharp"] \\ W^D \times C \arrow[r, "(j^*)^\sharp"] & W \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[r, "k^\flat"] \arrow[d, "(\mathrm{id}_Z\times j)^\flat"'] & W^D \arrow[d, "j^*"] \\ (Z\times D)^C \arrow[r, "(k^\sharp)_*"] & W^C \end{tikzcd}$$

Proof. By Lemma 1.23 it suffices to show that either diagram commutes in order to show they both commute. We will show the left diagram commutes. Recall that by how the pullback j^* is defined, that $(j^*)^\sharp = \varepsilon_W \circ (\mathrm{id}_{W^D \times j})$. Thus, it suffices to show the following diagram commutes:

(3)
$$\begin{tikzcd} Z \times C \arrow[rr, "\mathrm{id}\times j"] \arrow[d, "k^\flat\times\mathrm{id}"'] & & Z \times D \arrow[d, "k^\sharp"] \\ W^D \times C \arrow[r, "\mathrm{id}\times j"] & W^D \times D \arrow[r, "\varepsilon_D"] & W \end{tikzcd}$$

It is straightforward to see by the universal property of the product that $(\mathrm{id}_{W^D} \times j) \circ (k^\flat \times \mathrm{id}_C) = (k^\flat \times \mathrm{id}_D) \circ (\mathrm{id}_Z \times j)$. Then by Lemma 1.23 applied to the following diagram

$$\begin{tikzcd}
Z \arrow[r, "k^\flat"] \arrow[d, "k^\flat"'] & W^D \arrow[d, equal] \\
W^D \arrow[r, equal] & W^D
\end{tikzcd}$$

we get that $\varepsilon_D \circ (k^\flat \times \mathrm{id}_D) = (\mathrm{id}_{W^D})^\flat \circ (k^\flat \times \mathrm{id}_D) = \mathrm{id}_W \circ k^\sharp = k^\sharp$. To summarize, we get that

$$\varepsilon_D \circ (\mathrm{id}_{W^D} \times j) \circ (k^\flat \times \mathrm{id}_C) = \varepsilon_D \circ (k^\flat \times \mathrm{id}_D) \circ (\mathrm{id}_Z \times j) = k^\sharp \circ (\mathrm{id}_Z \times j),$$

so diagram (3) commutes, as desired. □

Now we prove the proposition.

Proof. Unravelling definitions, a lifting problem of the form (1) amounts to the data of maps $f^\sharp : A \times D \to X$, $g^\sharp : B \times C \to X$, and $h^\sharp : B \times D \to Y$ such that the following three diagrams commute:

$$\tag{4} \begin{tikzcd}
A \times C \arrow[r, "i \times \mathrm{id}_C"] \arrow[d, "\mathrm{id}_A \times j"'] & B \times C \arrow[d, "g^\sharp"] \\
A \times D \arrow[r, "f^\sharp"] & X
\end{tikzcd} \qquad \begin{tikzcd}
A \times D \arrow[r, "f^\sharp"] \arrow[d, "i \times \mathrm{id}_D"'] & X \arrow[d, "p"] \\
B \times D \arrow[r, "h^\sharp"] & Y
\end{tikzcd} \qquad \begin{tikzcd}
B \times C \arrow[r, "g^\sharp"] \arrow[d, "\mathrm{id}_B \times j"'] & X \arrow[d, "p"] \\
B \times D \arrow[r, "h^\sharp"] & Y
\end{tikzcd}$$

(The left diagram is the data of a morphism $A \times D \coprod_{A \times C} B \times C \to X$ and the other two diagrams assert commutativity of the lifting problem). We label these diagrams (4A), (4B), and (4C), respectively. In terms of these data, a solution to the lifting problem is a single arrow $\ell^\sharp : B \times D \to X$ which serves as a lift for both the diagrams (4B) and (4C).

Conversely, a lifting problem of the form (2) is the data of maps $f^\flat : A \to X^D$, $g^\flat : B \to X^C$, and $h^\flat : B \to Y^D$ such that the following three diagrams commute:

$$\tag{5} \begin{tikzcd}
B \arrow[r, "g^\flat"] \arrow[d, "h^\flat"'] & X^C \arrow[d, "p_*"] \\
Y^D \arrow[r, "j^*"] & Y^C
\end{tikzcd} \qquad \begin{tikzcd}
A \arrow[r, "f^\flat"] \arrow[d, "i"'] & X^D \arrow[d, "p_*"] \\
B \arrow[r, "h^\flat"] & Y^D
\end{tikzcd} \qquad \begin{tikzcd}
A \arrow[r, "f^\flat"] \arrow[d, "i"'] & X^D \arrow[d, "j^*"] \\
B \arrow[r, "g^\flat"] & X^C
\end{tikzcd}$$

(The left diagram is the data of a morphism $B \to X^C \times_{Y^C} Y^D$ and the other two diagrams asser commutativity of the lifting problem). We label these diagrams (5A), (5B), and (5C), respectively. In terms of these data, a solution to the lifting problem is a single arrow $\ell^\flat : B \to X^D$ which serves as a lift for both the diagrams (5B) and (5C).

Thus, in order to show the desired statement it suffices to show given arrows $f^\sharp : A \times D \to X$, $g^\sharp : B \times C \to X$, $h^\sharp : B \times D \to Y$, and $\ell^\sharp : B \times D \to X$, that:

- (4B) commutes iff (5B) commutes,
- (4A) commutes iff (5C) commutes,
- (4C) commutes iff (5A) commutes,
- $\ell^\sharp$ is a lift of (4B) iff $\ell^\flat$ is a lift of (5B), and
- Assuming $\ell^\sharp$ is a lift of $(4B)$ and $\ell^\flat$ is a lift of $(5B)$, $\ell^\sharp$ is a lift of (4C) iff $\ell^\flat$ is a lift of (5C).

To start with, note that (4B) commutes iff (5B) commutes, by Lemma 1.23.

Next, we claim that (4A) commutes iff (5C) commutes. By Lemma 1.23, (4A) commutes iff the following diagram commutes

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "(\mathrm{id}_A \times j)^\flat"'] & B \arrow[d, "g^\flat"] \\
(A \times D)^C \arrow[r, "(f^\sharp)_*"] & X^C
\end{tikzcd}$$

By Lemma 1.24 (the second diagram), $(f^\sharp)_* \circ (\mathrm{id}_A \times j)^\flat = j^* \circ f^\flat$, so this diagram commutes iff the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "f^\flat"'] & B \arrow[d, "g^\flat"] \\
X^D \arrow[r, "j^*"] & X^C
\end{tikzcd}$$

This is precisely (5C) (but flipped), as desired.

Next, we claim that (4C) commutes iff (5A) commutes. By Lemma 1.23, (5A) commutes iff the following diagram commutes

$$\begin{tikzcd}
B \times C \arrow[r, "g^\sharp"] \arrow[d, "h^\flat \times \mathrm{id}"'] & X \arrow[d, "p"] \\
Y^D \times C \arrow[r, "(j^*)^\sharp"] & Y
\end{tikzcd}$$

By Lemma 1.24 (the first diagram), $(j^*)^\sharp \circ (h^\flat \times \mathrm{id}_C) = h^\sharp \circ (\mathrm{id}_B \times j)$, so this diagram commutes iff the following diagram commutes:

$$\begin{tikzcd}
B \times C \arrow[r, "g^\sharp"] \arrow[d, "\mathrm{id}_B \times j"'] & X \arrow[d, "p"] \\
B \times D \arrow[r, "h^\sharp"] & Y
\end{tikzcd}$$

This is precisely (4C), as desired.

Now, we claim that ℓ is a lift for (4B) iff it is a lift for (5B). Indeed, by Lemma 1.23, $f^\sharp = \ell^\sharp \circ (i \times \mathrm{id}_D)$ iff $f^\flat = \ell^\flat \circ i$ and $p \circ \ell^\sharp = h^\sharp$ iff $p_* \circ \ell^\flat = h^\flat$:

$$\begin{tikzcd}
A \times D \arrow[r, "f^\sharp"] \arrow[d, "i \times \mathrm{id}_D"'] & X \arrow[d, equal] & {} \arrow[r, phantom, "\leftrightsquigarrow"] & {} & A \arrow[r, "f^\flat"] \arrow[d, "i"'] & X^D \arrow[d, equal] & B \times D \arrow[r, "\ell^\sharp"] \arrow[d, equal] & X \arrow[d, "p"] & {} \arrow[r, phantom, "\leftrightsquigarrow"] & {} & B \arrow[r, "\ell^\flat"] \arrow[d, equal] & X^D \arrow[d, "p_*"] \\
B \times D \arrow[r, "\ell^\sharp"] & X & & & B \arrow[r, "\ell^\flat"] & X^D & B \times D \arrow[r, "h^\sharp"] & Y & & & B \arrow[r, "h^\flat"] & Y^D
\end{tikzcd}$$

In other words, $\ell^\sharp$ makes the top (resp. bottom) triangle of (4B) commute iff $\ell^\flat$ makes the top (resp. bottom) triangle of (5B) commute, as desired.

Finally, it remains to show that if $\ell^\sharp$ and $\ell^\flat$ determine lifts of (4B) and (5B), respectively, then $\ell^\sharp$ is a lift for (4C) iff it is a lift for (5C). First of all, since $\ell^\sharp$ and $\ell^\flat$ define lifts of (4B) and (5B), we already have $p \circ \ell^\sharp = h^\sharp$ and $\ell^\flat \circ i = f^\flat$, so it is sufficient (and necessary) to show that $\ell^\sharp \circ (\mathrm{id}_B \times j) = g^\sharp$ iff $j^* \circ \ell^\flat = g^\flat$. Note by Lemma 1.24 (second diagram), $j^* \circ \ell^\flat = (\ell^\sharp)_* \circ (\mathrm{id}_Z \times j)^\flat$, so it suffices to show that $\ell^\sharp \circ (\mathrm{id}_B \times j) = g^\sharp$ iff $(\ell^\sharp)_* \circ (\mathrm{id}_Z \times j)^\flat = g^\flat$. Indeed, this follows by

Lemma 1.23:

$$\begin{tikzcd}
B \times C \arrow[r, equal] \arrow[d, "\mathrm{id}_B \times j"'] & B \times C \arrow[d, "g^\sharp"] & B \arrow[r, equal] \arrow[d, "(\mathrm{id}_B \times j)^\flat"'] & B \arrow[d, "g^\flat"] \\
B \times D \arrow[r, "\ell^\sharp"] & X & (B \times D)^C \arrow[r, "(\ell^\sharp)_*"] & X^C
\arrow[from=1-2, to=2-3, phantom, "\rightsquigarrow"]
\end{tikzcd} \qquad \square$$

2. The Model Structure on Topological Spaces

A map $f : X \to Y$ in **Top** is an *inclusion* if it is continuous, injective, and for all $U \subseteq X$ open, there is some $V \subseteq Y$ open such that $f^{-1}(V) = U$. If f is a closed inclusion and every point in $Y \setminus f(X)$ is closed, then we call f a *closed* T_1 *inclusion.* We will let $\mathcal{T}$ denote the class of closed T_1 inclusions in **Top**.

The symbol D^n will denote the unit disk in $\mathbb{R}^n$, and the symbol S^{n-1} will denote the unit sphere in $\mathbb{R}^n$, so that we have the boundary inclusions $S^{n-1} \hookrightarrow D^n$. In particular, for $n = 0$ we let $D^0 = \{0\}$ and $S^{-1} = \emptyset$.

Recall: If $F : \mathscr{J} \to \mathbf{Top}$ is a functor, where $\mathscr{J}$ is a small category, the limit of F is obtained by taking the limit in the category of sets, and then topologizing it with the *initial topology*, where if $\eta : \underline{\lim F} \Rightarrow F$ is the limit cone, then the topology on $\lim F$ is that with subbasis given by sets of the form $\eta_j^{-1}(U)$ where $j \in \mathscr{J}$ and $U \subseteq F_j$ is open. Similarly, the colimit of F is obtained by taking the colimit $\operatorname{colim} F$ in the category of sets and endowing it with the *final topology*, where a set $U \subseteq \operatorname{colim} F$ is open if and only if $\varepsilon_j^{-1}(U)$ is open in F_j for all $j \in \mathscr{J}$, where $\varepsilon : F \Rightarrow \underline{\operatorname{colim} F}$ is the colimit cone (equivalently, a set $C \subseteq \operatorname{colim} F$ is closed if and only if $\varepsilon_j^{-1}(C)$ is closed in F_j for all $j \in \mathscr{J}$).

Given a space X, we construct a functor $(-)^X : \mathbf{Top} \to \mathbf{Top}$ as follows: Given a space Y, define Y^X to be the space whose underlying set is the set $\mathbf{Top}(X, Y)$ of continuous maps $X \to Y$, and the topology on Y^X is the *compact-open topology*, i.e., the topology with subbasis given by the sets of the form

$$S(K, U) := \{f \in \mathbf{Top}(X, Y) : f(K) \subseteq U\}$$

for $K \subseteq X$ compact and $U \subseteq Z$ open. Given a continuous map $f : Y \to Z$, define the induced map $f_* : Y^X \to Z^X$ by $f_*(g) := f \circ g$. Unravelling definitions, we have that given $f : Y \to Z$ continuous, $f_*^{-1}(S(K, U)) = S(K, f^{-1}(U))$ for all $K \subseteq X$ compact and $U \subseteq Z$ open, so that f_* is continuous. Furthermore, $(-)^X$ is clearly functorial, by associativity and unitality of function composition.

Given a topological space X, we say that X is *locally compact* if for all points $x \in X$ and open neighborhoods U of x, there exists an open set $V \subseteq X$ with $x \in V$, $\overline{V} \subseteq U$, and $\overline{V}$ compact. We claim that $(-)^X$ is right adjoint to $- \times X$ when X is locally compact and Hausdorff.

Proposition 2.1. *If X is a locally compact Hausdorff space, then functor $- \times X$ is left adjoint to $(-)^X$ (so that in particular $- \times X$ preserves colimits).*

Proof. We start by constructing the counit and unit of the adjunction. Given a space Z, define the counit $\varepsilon_Z : X \times Z^X \to Z$ to be the evaluation function, taking a pair $(x, f) \mapsto f(x)$. First, we claim ε_Z is continuous. Suppose we are given an open set $V \subseteq Z$ and a point $(x, f) \in \varepsilon_Z^{-1}(V)$ (so $f(x) \in V$). Since f is continuous and X is locally compact, there exists an open set $U \subseteq X$ containing x such that $x \in U \subseteq \overline{U} \subseteq f^{-1}(V)$ with $\overline{U}$ compact. Then consider the open set $U \times S(\overline{U}, V)$ in $X \times Y^X$. First of all, $(x, f) \in U \times S(\overline{U}, V)$, as $x \in U$ and $\overline{U} \subseteq f^{-1}(V)$, so that $f(\overline{U}) \subseteq V$ meaning $f \in S(\overline{U}, V)$. Furthermore, given $(y, g) \in U \times S(\overline{U}, V)$, we have $\varepsilon_Z(y, g) = g(y) \in g(U) \subseteq g(\overline{U}) \subseteq V$, so $U \times S(\overline{U}, V)$ is an open neighborhood of x contained in $\varepsilon_Z^{-1}(V)$, as desired. Hence, ε_Z is continuous. It remains to show naturality. Given a map

$f : Z \to W$, we wish to show the following diagram commutes:

$$\begin{tikzcd}
X \times Z^X \arrow[r, "\varepsilon_Z"] \arrow[d, "\mathrm{id}_X \times f_*"'] & Z \arrow[d, "f"] \\
X \times W^X \arrow[r, "\varepsilon_W"] & W
\end{tikzcd}$$

Indeed, chasing an element (x, g) around the diagram yields:

$$\begin{tikzcd}
(x, g) \arrow[r, mapsto] \arrow[d, mapsto] & g(x) \arrow[d, mapsto] \\
(x, f \circ g) \arrow[r, mapsto] & f(g(x))
\end{tikzcd}$$

so it does indeed commute.

Now we wish to define the unit $\eta_Y : Y \to (Y \times X)^X$. Given $y \in Y$, define $\eta_Y(y) \in (Y \times X)^X$ by $\eta_Y(y)(x) := (y, x)$. First of all, for it to be true that $\eta_Y(y) \in (X \times Y)^X$, it must be true that $\eta_Y(y)$ is continuous. Indeed, this is clear as η_Y is obtained as the product map $y \times \mathrm{id}_X : X \to Y \times X$, where y represents the constant function on y (which is obviously continuous). Furthermore, η_Y itself is continuous: given $K \subseteq X$ compact and $U \subseteq Y \times X$ open, we wish to show that $\eta_Y^{-1}(S(K, U))$ is open in Y. It suffices to show that given $y \in \eta_Y^{-1}(S(K, U))$, there exists an open neighborhood W of y that is mapped by η_Y into $S(K, U)$. Since $y \in \eta_Y^{-1}(S(K, U))$, $\eta_Y(y)(K) = \{y\} \times K \subseteq U$. Then $U \cap (Y \times K)$ is an open set in the subspace $Y \times K$ containing the slice $\{y\} \times K$. By definition of the product topology, for each $k \in K$, there exist open sets $W_k \subseteq Y$ and $V_k \subseteq K$ such that $(y, k) \in W_k \times V_k \subseteq U \cap (Y \times K)$. Then the V_k's form an open cover of K, which is compact, so that there exist $k_1, \ldots, k_n \in K$ with $V_{k_1} \cup \cdots \cup V_{k_n} = K$. Hence if we define $W := W_{k_1} \cap \cdots \cap W_{k_n}$, then $\{y\} \times K \subseteq W \times K \subseteq U \cap (Y \times K)$, and W is open in Y as it is a finite intersection of open sets. Then for all $w \in W$, $\eta_Y(w)(K) = \{w\} \times K \subseteq W \times K \subseteq U$. Hence, indeed η_Y is continuous. It remains to show naturality. Given a map $f : Y \to W$, we wish to show the following diagram commutes:

$$\begin{tikzcd}
Y \arrow[r, "\eta_Y"] \arrow[d, "f"'] & (Y \times X)^X \arrow[d, "(f \times \mathrm{id}_X)_*"] \\
W \arrow[r, "\eta_W"] & (W \times X)^X
\end{tikzcd}$$

Indeed, chasing an element y around the top of the diagram yields the function obtained as the composition $x \mapsto (y, x) \mapsto f \times \mathrm{id}_X(y, x) = (f(y), x)$, while chasing around the bottom of the diagram more directly yields the function $x \mapsto (f(y), x)$.

Now that we have constructed the unit and counit, it remains to verify the counit-unit equations, i.e., that for each $Y \in \mathbf{Top}$ that $\varepsilon_{Y\times X} \circ (\eta_Y \times \mathrm{id}_X) = \mathrm{id}_{Y\times X}$ and $(\varepsilon_Y)_* \circ \eta_{Y^X} = \mathrm{id}_{Y^X}$. First of all, given $(y, x) \in Y \times X$, we have

$$(\varepsilon_{Y\times X} \circ (\eta_Y \times \mathrm{id}_X))(y, x) = \varepsilon_{X\times Y}(\eta_Y(y), x) = \eta_Y(y)(x) = (y, x).$$

On the other hand, given $f \in Y^X$, we have

$$(\varepsilon_Y)_*(\eta_{Y^X}(f)) = (\varepsilon_Y)_*([x \mapsto (f, x)]) = [x \mapsto (f, x) \mapsto \varepsilon_Y(f, x) = f(x)] = f.$$

Hence, indeed ε and η form the counit and unit for the adjoint pair $(- \times X, (-)^X)$. □

Now that we have gotten some topological preliminaries out of the way, we are ready to define the model structure.

Definition 2.2. A map $f : X \to Y$ in **Top** is called a *weak equivalence* if

$$\pi_n(f, x) : \pi_n(X, x) \to \pi_n(Y, f(x))$$

is an isomorphism for all $n \geq 0$ and for all $x \in X$. We will write $\mathcal{W}$ to refer to the class of all weak equivalences in **Top**.

Define the set of maps I' to consist of all the boundary inclusion $S^{n-1} \hookrightarrow D^n$ for all $n \geq 0$, and define the set J to consist of all the inclusions $D^n \hookrightarrow D^n \times I$ mapping $x \mapsto (x, 0)$ for $n \geq 0$. Then a map f will be called a *cofibration* if it is in I'-cof $= {}_\perp(I'_\perp)$, and a *fibration* if it is in J-inj $= J_\perp$.

A map in I'-cell is usually called a *relative cell complex*; a relative CW-complex is a special case of a relative cell complex, where, in particular, the cells can be attached in order of their dimension. Note that in particular maps of J are relative CW complexes, hence are relative I'-cell complexes. A fibration is often known as a *Serre fibration* in the literature.

Theorem 2.3 (Hovey Theorem 2.4.19)**.** *There is a finitely generated model structure on* **Top** *with I' as the set of generating cofibrations, J as the set of generating trivial cofibrations, and the cofibrations, fibrations, and weak equivalences as above. Every object of* **Top** *is fibrant, and the cofibrant objects are retracts of relative cell complexes.*

Proof. We will apply Theorem 1.21 to get that there is a cofibrantly generated model structure on **Top** with I' as the set of generating cofibrations, J as the set of generating trivial fibrations, and $\mathcal{W}$ as the subcategory of weak equivalences. The six requirements outlined in the theorem will be verified like so:

1. $\mathcal{W}$ is a subcategory of $\mathcal{C}$ which has the 2-of-3 property and is closed under retracts: Lemma 2.12.
2. The domains of I' are small relative to I'-cell: Proposition 2.11.
3. The domains of J are small relative to J-cell: Proposition 2.11.
4. J-cell $\subseteq \mathcal{W} \cap {}_\perp(I'_\perp)$: In Proposition 2.13, we will show ${}_\perp(J_\perp) \subseteq \mathcal{W} \cap {}_\perp(I'_\perp)$, and by Corollary 1.16 J-cell $\subseteq {}_\perp(J_\perp)$.
5. $I'_\perp \subseteq \mathcal{W} \cap J_\perp$: Proposition 2.14
6. $\mathcal{W} \cap J_\perp \subseteq I'_\perp$: Proposition 2.24

It will follow by the definition of a cofibrantly generated model structure (Definition 1.19) that the fibrations in this model structure are given by $J_\perp$, which is precisely how we defined it. By Proposition 1.20, the class of cofibrations will be given by ${}_\perp(I'_\perp)$, which is likewise exactly how we defined them.

In Proposition 2.8, we will show that compact spaces are finite relative to the class $\mathcal{T}$ of closed T_1 inclusions. Hence, this model structure will be finitely generated, as the domains and codomains of I' and J are all compact, and by the reasoning given above we will have shown I'-cell $\subseteq \mathcal{T}$.

We will show that every object of **Top** is fibrant in Corollary 2.17. □

Lemma 2.4. *Let λ be an ordinal, and X a λ-sequence in* **Top**. *Then:*

- *(i) If X is a λ-sequence of injections, then $X_\alpha \to X_\beta$ is an injective for all $\alpha \leq \beta < \lambda$.*
- *(ii) If X is a λ-sequence of inclusions, then the map $X_\alpha \to X_\beta$ is an inclusion for all $\alpha \leq \beta < \lambda$.*
- *(iii) If X is a λ-sequence of closed T_1 inclusions, then the map $X_\alpha \to X_\beta$ is a closed T_1 inclusion for all $\alpha \leq \beta < \lambda$.*

Proof. In what follows, given $\alpha \leq \beta < \lambda$, let $\iota_{\alpha,\beta}$ denote the map $X_\alpha \to X_\beta$.

- (i) Let $\alpha < \lambda$. We perform a proof by transfinite induction on β for $\alpha \leq \beta < \lambda$ that $\iota_{\alpha,\beta} : X_\alpha \to X_\beta$ is injective. For the zero case, clearly $\iota_{\alpha,\alpha} = \mathrm{id}_{X_\alpha}$ is injective. Supposing $\iota_{\alpha,\beta}$ is injective for some $\alpha < \beta + 1 < \lambda$, we have $\iota_{\alpha,\beta+1} = \iota_{\beta,\beta+1} \circ \iota_{\alpha,\beta}$ is a composition of injections, and is therefore clearly injective itself. Finally, suppose γ is a limit ordinal with $\alpha \leq \gamma < \lambda$ such that $\iota_{\alpha,\beta}$ is injective for all $\alpha \leq \beta < \gamma$. We claim $\iota_{\alpha,\gamma}$ is injective. Since X_γ is colimit preserving and γ is a limit ordinal, X_γ is the colimit of the diagram $\{X_\beta\}_{\beta<\gamma}$ via the maps $\iota_{\beta,\gamma}$, so that in particular by Remark 1.7 and the fact that the forgetful functor **Top** $\to$ **Set** preserves colimits, given $a, b \in X_\alpha$ with $\iota_{\alpha,\gamma}(a) = \iota_{\alpha,\gamma}(b)$, there exists some $\beta < \gamma$ with $\iota_{\alpha,\beta}(a) = \iota_{\alpha,\beta}(b)$, and $\iota_{\alpha,\beta}$ is injective for all $\beta < \gamma$, so it must have been true $a = b$ in X_α.

(ii) By part(i), we know that $\iota_{\alpha,\beta}$ is injective for $\alpha \leq \beta < \lambda$. Thus it suffices to prove the following statement: For all $\alpha < \lambda$ and $U \subseteq X_\alpha$, for all $\alpha \leq \beta < \lambda$, there exists $U_\beta \subseteq X_\beta$ with $U_\alpha = U$ such that for all $\alpha \leq \beta' \leq \beta < \lambda$, $\iota^{-1}_{\beta',\beta}(U_\beta) = U_{\beta'}$. We prove this by transfinite recursion on $\alpha \leq \beta < \lambda$.

The zero case has been taken care of: $U_\alpha = U$. For the sucessor case, given $\alpha < \beta+1 < \lambda$, supposing U_β has been defined with the desired properties, since $\iota_{\beta,\beta+1}$ is an inclusion, there exists $U_{\beta+1} \subseteq X_{\beta+1}$ with $\iota^{-1}_{\beta,\beta+1}(U_{\beta+1}) = U_\beta$. Then given $\alpha \leq \beta' \leq \beta+1$, we have

$$\iota^{-1}_{\beta',\beta+1}(U_{\beta+1}) = (\iota_{\beta,\beta+1} \circ \iota_{\beta',\beta})^{-1}(U_{\beta+1}) = \iota^{-1}_{\beta',\beta}(\iota^{-1}_{\beta,\beta+1}(U_{\beta+1})) = \iota^{-1}_{\beta',\beta}(U_\beta) = U_{\beta'}.$$

Finally, the limit case. Suppose γ is a limit ordinal with $\alpha < \gamma \leq \lambda$, and suppose U_β has been constructed with the desired properties for $\alpha \leq \beta < \gamma$. We wish to define U_γ. Since X is colimit preserving and $\gamma = \sup_{\alpha \leq \beta < \gamma} \beta$, the maps $\iota_{\beta,\gamma}$ for $\alpha \leq \beta < \gamma$ form a colimit cone for the diagram $\{X_\beta\}_{\alpha \leq \beta < \gamma}$. Let $S = \{0, 1\}$ be the Sierpinski space whose open sets are $\{\emptyset, \{1\}, \{0, 1\}\}$. For $\alpha \leq \beta < \gamma$, define a map $s_\beta : X_\beta \to S$ mapping everything in U_β to 1 and every other point to 0. Each s_β is clearly continuous, as $s^{-1}_\beta(1) = U_\beta$. Furthermore, we claim the s_β's form a cone under the diagram $\{X_\beta\}_{\alpha \leq \beta < \gamma}$, i.e., that given $\alpha \leq \beta' \leq \beta < \gamma$, the following diagram commutes

$$\begin{tikzcd}
X_{\beta'} \arrow[rr, "\iota_{\beta',\beta}"] \arrow[dr, "s_{\beta'}"'] & & X_\beta \arrow[dl, "s_\beta"] \\
& S &
\end{tikzcd}$$

To see this, let $x \in X_{\beta'}$. If $x \in U_{\beta'} = \iota^{-1}_{\beta',\beta}(U_\beta)$, then $\iota_{\beta',\beta}(x) \in U_\beta$, so $s_\beta(\iota_{\beta',\beta}(x)) = 1 = s_{\beta'}(x)$. Conversely, if $x \in X_{\beta'} \setminus U_{\beta'} = X_{\beta'} \setminus \iota^{-1}_{\beta',\beta}(U_\beta)$, then $x \notin \iota^{-1}_{\beta',\beta}(U_\beta)$, so $\iota_{\beta',\beta}(x) \notin U_\beta$, meaning $s_\beta(\iota_{\beta',\beta}(x)) = 0 = s_{\beta'}(0)$. Hence, the s_β's do indeed form a cone under $\{X_\beta\}_{\alpha \leq \beta < \gamma}$, so by universal property of the colimit there exists a unique map $\ell : X_\gamma \to S$ such that $s_\beta = \ell \circ \iota_{\beta,\gamma}$ for all $\alpha \leq \beta < \gamma$. Define $U_\gamma := \ell^{-1}(1)$, which is open as $\{1\}$ is open in S. It remains to show that for all $\alpha \leq \beta \leq \gamma$ that $\iota^{-1}_{\beta,\gamma}(U_\gamma) = U_\beta$. Indeed, we have

$$\iota^{-1}_{\beta,\gamma}(U_\gamma) = \iota^{-1}_{\beta,\gamma}(\ell^{-1}(1)) = (\ell \circ \iota_{\beta,\gamma})^{-1}(1) = s^{-1}_\beta(1) = U_\beta.$$

(iii) By part (ii), we know that $\iota_{\alpha,\beta}$ is an inclusion for $\alpha \leq \beta < \lambda$. Fix $\alpha < \lambda$. We perform transfinite induction on $\alpha \leq \beta < \lambda$ to show that $\iota_{\alpha,\beta}$ is a closed T_1 inclusion, assuming it is already an inclusion. For the zero case, clearly $\iota_{\alpha,\alpha} = \mathrm{id}_{X_\alpha}$ is closed, and vacuosuly very point in $X_\alpha \setminus \iota_{\alpha,\alpha}(X_\alpha) = \emptyset$ is a closed point. For the successor case, supposing $\iota_{\alpha,\beta} : X_\alpha \to X_\beta$ is a closed T_1 inclusion, we wish to show that $\iota_{\alpha,\beta+1} : X_\alpha \to X_{\beta+1}$ is a closed T_1 inclusion. Since $\iota_{\alpha,\beta+1} = \iota_{\beta,\beta+1} \circ \iota_{\alpha,\beta}$ is a composition of closed T_1 inclusions, it is clearly closed. It remains to show that every point in $X_{\beta+1} \setminus \iota_{\alpha,\beta+1}(X_\alpha)$ is closed in $X_{\beta+1}$. Indeed, let $x \in X_{\beta+1} \setminus \iota_{\alpha,\beta+1}(X_\alpha)$. First, if $x \in X_{\beta+1} \setminus \iota_{\beta,\beta+1}(X_\beta)$, we are done, as $\iota_{\beta,\beta+1}$ is a closed T_1 inclusion. Hence, we may assume that $x \in \iota_{\beta,\beta+1}(X_\beta)$, so there exists some $y \in X_\beta$ such that $\iota_{\beta,\beta+1}(y) = x$. Since $\iota_{\beta,\beta+1}$ is closed, in order to show x is a closed point in $X_{\beta+1}$, it suffices to show that y is a closed point in X_β. Since $\iota_{\alpha,\beta}$ is a closed T_1 inclusion, it further suffices to show that y is not in the image of $\iota_{\alpha,\beta}$. Suppose for the sake of a contradiction that there existed $z \in X_\alpha$ with $\iota_{\alpha,\beta}(z) = y$. Then we would have

$$\iota_{\alpha,\beta+1}(z) = \iota_{\beta,\beta+1}(\iota_{\alpha,\beta}(z)) = \iota_{\beta,\beta+1}(y) = x,$$

a contradiction of the fact that $x \in X_{\beta+1} \setminus \iota_{\alpha,\beta+1}(X_\alpha)$. Hence, it must have been true that y is not in the image of $\iota_{\alpha,\beta}$ in the first place, the desired result. Finally, the limit case. Suppose γ is a limit ordinal with $\alpha < \gamma \leq \lambda$ such that $\iota_{\alpha,\beta}$ is a closed T_1 inclusion for all $\alpha \leq \beta < \gamma$. Then we wish to show $\iota_{\alpha,\gamma}$ is a closed T_1 inclusion.

First, we show $\iota_{\alpha,\gamma}$ is closed. Let $C \subseteq X_\alpha$ be closed. Since $\gamma = \sup_{\alpha \leq \beta < \gamma} \beta$ and X is colimit-preserving, X_γ is the colimit of the X_β's for $\alpha \leq \beta < \gamma$ via the maps $\iota_{\beta,\gamma}$, and the topology on X_γ is the final topology induced by these maps. Hence, in order to show $\iota_{\alpha,\gamma}(C)$ is closed in X_γ, it suffices to show that $\iota_{\beta,\gamma}^{-1}(\iota_{\alpha,\gamma}(C))$ is closed in X_β for all $\alpha \leq \beta < \gamma$. It further suffices to show that $\iota_{\beta,\gamma}^{-1}(\iota_{\alpha,\gamma}(C)) = \iota_{\alpha,\beta}(C)$, as $\iota_{\alpha,\beta}$ is closed. First, suppose $x \in \iota_{\beta,\gamma}^{-1}(\iota_{\alpha,\gamma}(C))$, so $\iota_{\beta,\gamma}(x) = \iota_{\alpha,\gamma}(c)$ for some $c \in C$. Then since the forgetful functor **Top** $\to$ **Set** preserves colimits, by the explicit description of the colimit in **Set** (Remark 1.7), there exists μ with $\alpha, \beta \leq \mu < \gamma$ such that $\iota_{\beta,\mu}(x) = \iota_{\alpha,\mu}(c)$. But $\iota_{\alpha,\mu} = \iota_{\beta,\mu} \circ \iota_{\alpha,\beta}$, and $\iota_{\beta,\mu}$ is injective (by (i)) so $x = \iota_{\alpha,\beta}(c)$, meaning $x \in \iota_{\alpha,\beta}(C)$, as desired. Conversely, suppose we are given $c \in C$, then we wish to show $\iota_{\alpha,\beta}(c) \in \iota_{\beta,\gamma}^{-1}(\iota_{\alpha,\gamma}(C))$, i.e., that $\iota_{\beta,\gamma}(\iota_{\alpha,\beta}(c)) \in \iota_{\alpha,\gamma(C)}$. This follows immediately as $\iota_{\beta,\gamma} \circ \iota_{\alpha,\beta} = \iota_{\alpha,\gamma}$.

Lastly, we show that for all $x \in X_\gamma \setminus \iota_{\alpha,\gamma}(X_\alpha)$ that x is a closed point in X_γ. Again by the description of the colimit in **Set** (Remark 1.7), the fact that the forgetful functor **Top** $\to$ **Set** preserves colimits, and that X preserves colimits, we know that every point in X_γ is in the image of some $\iota_{\beta,\gamma}$ for some $\alpha \leq \beta < \gamma$. Hence, there exists some $\alpha < \beta < \gamma$ and a point $y \in X_\beta$ with $\iota_{\beta,\gamma}(y) = x$. By the preceeding paragraph, $\iota_{\beta,\gamma}$ is closed, so in order to show x is a closed point in X_γ it suffices to show that y is a closed point in X_β. It further suffices to show that $y \in X_\beta \setminus \iota_{\alpha,\beta}(X_\alpha)$, as $\iota_{\alpha,\beta}$ is a closed T_1 inclusion. Suppose for the sake of a contradiction that there existed some $z \in X_\alpha$ such that $\iota_{\alpha,\beta}(z) = y$. Then we would have

$$\iota_{\alpha,\gamma}(z) = \iota_{\beta,\gamma}(\iota_{\alpha,\beta}(z)) = \iota_{\beta,\gamma}(y) = x,$$

a contradiction of the fact that $x \in X_\gamma \setminus \iota_{\alpha,\gamma}(X_\alpha)$. Hence, y must not have been in the image of $\iota_{\alpha,\beta}$ in the first place, as desired. ☐

This result, by Lemma 1.2 and Lemma 1.3, gives the following corollaries:

Corollary 2.5. *The class of injective maps (resp. inclusions, closed T_1 inclusions) in* **Top** *is closed under transfinite composition.*

Corollary 2.6. *Let λ be an ordinal, and X be a λ-sequence in* **Top**. *Then:*

- *(i) If X is a λ-sequence of injections, then the canonical map $X_\alpha \to \operatorname{colim} X$ is an injection for all $\alpha < \lambda$.*
- *(ii) If X is a λ-sequence of inclusions, then the canonical map $X_\alpha \to \operatorname{colim} X$ is an inclusion for all $\alpha < \lambda$.*
- *(iii) If X is a λ-sequence of closed T_1 inclusions, then the canonical map $X_\alpha \to \operatorname{colim} X$ is a closed T_1 inclusion for all $\alpha < \lambda$.*

Lemma 2.7 (Hovey 2.4.1)**.** *Every topological space is small relative to the inclusions.*

Proof. We claim that every topological space A is $|A|$-small relative to the inclusions. We use the characterization of smallness afforded by Proposition 1.8. Let λ be an $|A|$-filtered ordinal, and let $X : \lambda \to$ **Top** be a λ-sequence so that $X_\beta \to X_{\beta+1}$ is an inclusion for all $\beta + 1 < \lambda$. Recall that the forgetful functor **Top** $\to$ **Set** is forgetful, so elements of $\operatorname{colim} X$ are equivalence classes of elements $a \in X_\alpha$ for $\alpha < \lambda$, where $a \in X_\alpha$ and $b \in X_\beta$ represent the same equivalence class iff there exists $\alpha, \beta \leq \gamma < \lambda$ so that a and b are sent to the same element by the maps $X_\alpha \to X_\gamma$ and $X_\beta \to X_\gamma$, respectively.

First, suppose $f : A \to X_\alpha$ and $g : A \to X_\beta$ are continuous maps such that the compositions $A \xrightarrow{f} X_\alpha \to \operatorname{colim} X$ and $A \xrightarrow{g} X_\beta \to \operatorname{colim} X$ are equal. Then the same proof given in Example 1.9 works to show that f and g are equal in some stage of the colimit, as desired.

Conversely, suppose we are given a (continuous) map $f : A \to \operatorname{colim} X$. As in the proof of Example 1.9, we may find some $\beta < \lambda$ and a map of sets $\widetilde{f} : A \to X_\beta$ such that the composition $A \xrightarrow{\widetilde{f}} X_\beta \xrightarrow{j} \operatorname{colim} X$ is equal to f (note we have given the canonical map $X_\beta \to \operatorname{colim} X$ the name j). It remains to show that $\widetilde{f}$ is continuous. Let $U \subseteq X_\beta$ be open. Since j is an inclusion (Corollary 2.6), there exists $V \subseteq \operatorname{colim} X_\beta$ open such that $j^{-1}(V) = U$. Then $\widetilde{f}^{-1}(U) = \widetilde{f}^{-1}(j^{-1}(V)) = (j \circ \widetilde{f})^{-1}(V) = f^{-1}(V)$, and f is continuous, so $\widetilde{f}^{-1}(U) = f^{-1}(V)$ is open. Thus $\widetilde{f}$ is continuous, as desired. □

Proposition 2.8 (Hovey 2.4.2)**.** *Compact topological spaces are finite relative to the class $\mathfrak{T}$ of closed T_1 inclusions.*

Proof. We use the characterization of smallness afforded by Proposition 1.8. Let λ be a limit ordinal, and let $X : \lambda \to \mathbf{Top}$ be a λ-sequence so that $X_\beta \to X_{\beta+1}$ is a closed T_1 inclusion for all $\beta + 1 < \lambda$. Let $j : X \Rightarrow \underline{\operatorname{colim} X}$ is a colimit cone for X. Recall that the forgetful functor $\mathbf{Top} \to \mathbf{Set}$ is forgetful, so by Remark 1.7 elements of $\operatorname{colim} X$ are equivalence classes of elements $a \in X_\alpha$ for $\alpha < \lambda$, where $a \in X_\alpha$ and $b \in X_\beta$ represent the same equivalence class iff there exists $\alpha, \beta \leq \gamma < \lambda$ so that a and b are sent to the same element by the maps $X_\alpha \to X_\gamma$ and $X_\beta \to X_\gamma$, respectively.

First, we show condition (i) of Proposition 1.8. Let $j : X \Rightarrow \underline{\operatorname{colim} X}$ be a colimt cone for X, and suppose we are given maps $f : A \to X_\alpha$ and $g : A \to X_\beta$ such that $j_\alpha \circ f = j_\beta \circ g$. WLOG, suppose $\alpha \leq \beta$. Then

$$j_\beta \circ \iota_{\alpha,\beta} \circ f = j_\alpha \circ f = j_\beta \circ g,$$

and j_β is injective (Corollary 2.6) and therefore a monomorphism in **Top**, so $\iota_{\alpha,\beta} \circ f = g$, meaning f and g do indeed agree in some stage of the colimit, as desired.

Now we show condition (ii) of Proposition 1.8. Let $f : A \to \operatorname{colim} X$ be a continuous map. In order to show f factors through some X_α, we first claim it is sufficient for there to be some $\alpha < \lambda$ with $f(A) \subseteq j_\alpha(X_\alpha)$. Given an ordinal $\alpha < \lambda$, for each $a \in A$, there exists $\widetilde{f}(a) \in X_\alpha$ such that $j_\alpha(\widetilde{f}(a)) = f(a)$. Thus we have defined a function $\widetilde{f} : A \to X_\alpha$ such that $j_\alpha \circ \widetilde{f} = f$. It remains to show that $\widetilde{f}$ is continuous. Indeed, we know j_α is an inclusion (Corollary 2.6), so given $U \subseteq X_\alpha$ open, there exists $V \subseteq \operatorname{colim} X$ open with $j_\alpha^{-1}(V) = U$, in which case

$$\widetilde{f}^{-1}(U) = \widetilde{f}^{-1}(j_\alpha^{-1}(V)) = (j_\alpha \circ \widetilde{f})^{-1}(V) = f^{-1}(V),$$

which is open as f is continuous. Hence, $\widetilde{f}$ is continuous, as desired.

Now, suppose for the sake of a contradiction that for all $\alpha < \lambda$, $f(A) \not\subseteq j_\alpha(X_\alpha)$. Thus we may construct a **strictly increasing** sequence $\{\alpha_n\}_{n=0}^\infty \subseteq \lambda$ such that for $n > 0$, there exists $x_n \in j_{\alpha_n}(X_{\alpha_n}) \setminus j_{\alpha_{n-1}}(X_{\alpha_{n-1}})$ with $x_n \in f(A)$. Thus for each $n > 0$, there exists $y_n \in X_{\alpha_n}$ such that $j_{\alpha_n}(y_n) = x_n$. Note in particular that given $0 \leq m < n$, y_n is not in the image of $\iota_{\alpha_m,\alpha_n}$. Suppose for the sake of a contradiction that $y_n = \iota_{\alpha_m,\alpha_n}(z)$ for some $z \in X_{\alpha_m}$ and $0 \leq m < n$. Then we know $j_{\alpha_m}(z) = j_{\alpha_n}(\iota_{\alpha_m,\alpha_n}(z)) = j_{\alpha_n}(y_n) = x_n$, and

$$x_n \in j_{\alpha_n}(X_{\alpha_n}) \setminus j_{\alpha_{n-1}}(X_{\alpha_{n-1}}) \supseteq j_{\alpha_n}(X_{\alpha_n}) \setminus j_{\alpha_{n-1}}(\iota_{\alpha_m,\alpha_{n-1}}(X_{\alpha_m})) = j_{\alpha_n}(X_{\alpha_n}) \setminus j_{\alpha_m}(X_{\alpha_m}).$$

Hence we reach a contradiction, as $j_{\alpha_m}(z) = x_m$ but x_n is not in the image of j_{α_m}. Let $\mu := \sup_{n=1}^\infty \alpha_n$. Clearly $\mu \leq \lambda$; if $\mu = \lambda$, define $X_\mu := \operatorname{colim} X$, $j_\mu := \mathrm{id}_{X_\mu}$, and for $\alpha < \lambda$ define $\iota_{\alpha,\mu} := j_\alpha$. Let $K := \{\iota_{\alpha_n,\mu}(y_n)\}_{n=1}^\infty \subseteq X_\mu$. We claim every subset of K is closed in X_μ. Since X is colimit preserving and $\mu = \sup_{n=1}^\infty \alpha_n$, the topology on X_μ is the final topology induced by the maps $\iota_{\alpha_n,\mu} : X_{\alpha_n} \to X_\mu$ for $n = 1, 2, \ldots$. Thus, given a subset $C \subseteq K$, in order to show that C is closed in X_μ, it is sufficient (and necessary) for $\iota_{\alpha_n,\mu}^{-1}(C)$ to be closed in X_{α_n} for $n = 1, 2, \ldots$. Let

$n > 0$. Given $y \in \iota_{\alpha_n,\mu}^{-1}(C)$, then $\iota_{\alpha_n,\mu}(y) \in C \subseteq K$, so that in particular $\iota_{\alpha_n,\mu}(y) = \iota_{\alpha_m,\mu}(y_m)$ for some $m = 1, 2, \ldots$. We claim $m \leq n$. Suppose for the sake of a contradiction that $m > n$, then we would have

$$\iota_{\alpha_m,\mu}(y_m) = \iota_{\alpha_n,\mu}(y) = \iota_{\alpha_m,\mu}(\iota_{\alpha_n,\alpha_m}(y)),$$

and $\iota_{\alpha_m,\mu}$ is injective (by either Lemma 2.4 if $\mu < \lambda$ or by Corollary 2.6 if $\mu = \lambda$, in which case recall we defined $\iota_{\alpha_m,\mu} = j_{\alpha_m}$), thus $y_m = \iota_{\alpha_n,\alpha_m}(y)$, meaning y_m is in the image of $\iota_{\alpha_n,\alpha_m}$ for $m > n$, a contradiction, as we showed earlier this is impossible. Thus it must have been true that $m \leq n$ in the first place, so

$$\iota_{\alpha_n,\mu}(y) \in \{\iota_{\alpha_m,\mu}(y_m)\}_{m=1}^{n} \implies y \in \iota_{\alpha_n,\mu}^{-1}(\{\iota_{\alpha_m,\mu}(y_m)\}_{m=1}^{n}).$$

We further claim $\iota_{\alpha_n,\mu}^{-1}(\{\iota_{\alpha_m,\mu}(y_m)\}_{m=1}^{n}) = \{\iota_{\alpha_m,\alpha_n}(y_m)\}_{m=1}^{n}$. To see the inclusion $\subseteq$, suppose $z \in X_{\alpha_n}$ with $\iota_{\alpha_n,\mu}(z) = \iota_{\alpha_m,\mu}(y_m)$ for some $m \leq n$. Then $\iota_{\alpha_n,\mu}(z) = \iota_{\alpha_n,\mu}(\iota_{\alpha_m,\alpha_n}(y_m))$ and $\iota_{\alpha_n,\mu}$ is injective (Lemma 2.4 if $\mu < \lambda$ and Corollary 2.6 if $\mu = \lambda$), so $z = \iota_{\alpha_m,\alpha_n}(y_m)$, as desired. To see the opposite inclusion, given $m \leq n$, we have $\iota_{\alpha_n,\mu}(\iota_{\alpha_m,\alpha_n}(y_m)) = \iota_{\alpha_m,\mu}(y_m)$, so $\iota_{\alpha_m,\alpha_n}(y_m) \in \iota_{\alpha_n,\mu}^{-1}(\{\iota_{\alpha_m,\mu}(y_m)\}_{m=1}^{n})$, as desired. Thus, we have shown $y \in \{\iota_{\alpha_m,\alpha_n}(y_m)\}_{m=1}^{n}$. Recall our choice of $y \in \iota_{\alpha_n,\mu}^{-1}(C)$ was arbitrary, so $\iota_{\alpha_n,\mu}^{-1}(C)$ is contained in $\{\iota_{\alpha_m,\alpha_n}(y_m)\}_{m=1}^{n}$. Thus, because $\{\iota_{\alpha_m,\alpha_n}(y_m)\}_{m=1}^{n}$ is finite, in order to show $\iota_{\alpha_n,\mu}^{-1}(C)$ is closed in X_{α_n}, it suffices to show that $\iota_{\alpha_m,\alpha_n}(y_m)$ is a closed point in X_{α_n} for $m = 1, \ldots, n$. As we have shown above, y_m is not in the image of $\iota_{\alpha_0,\alpha_m}$ for any $m \geq 1$, and $\iota_{\alpha_0,\alpha_m}$ is a closed T_1 inclusion (Lemma 2.4), so y_m is a closed point of X_{α_m} for $m = 1, \ldots, n$. Then since $\iota_{\alpha_m,\alpha_n}$ is closed (again by Lemma 2.4), $\iota_{\alpha_m,\alpha_n}(y_m)$ is closed in X_{α_n} for $m = 1, \ldots, n$, precisely the desired result.

Now, we have shown that every subset of K is closed in X_μ. Then $j_\mu : X_\mu \to \operatorname{colim} X$ is a closed and injective (this follows by Corollary 2.6 if $\mu < \lambda$, and if $\mu = \lambda$, $X_\mu = \operatorname{colim} X$, in which case j_μ is the identity), so every subset of $S := j_\mu(K)$ is closed in $\operatorname{colim} X$. Note that

$$S = \{j_\mu(\iota_{\alpha_n,\mu}(y_n))\}_{n=1}^{\infty} = \{j_{\alpha_n}(y_n)\}_{n=1}^{\infty} = \{x_n\}_{n=1}^{\infty} \subseteq f(A),$$

Then for $n = 1, 2, \ldots$, define $U_n := f(A) \setminus (S \setminus \{x_n\})$. Each U_n is open in $f(A)$ (as $S \setminus \{x_n\}$ is a subset of S and is therefore closed in $\operatorname{colim} X$, thus in $f(A)$), and the collection $\{U_n\}_{n=1}^{\infty}$ forms an infinite open cover of $f(A)$. Finally, this open cover has no finite subcover, as U_n is the only element of the cover containing x_n for $n = 1, 2, \ldots$. Hence we reach a contradiction, as f is continuous and A is compact, so $f(A)$ is compact, but we have found an infinite open cover of $f(A)$ which has no finite subcover. Thus, there must have existed soem $\alpha < \lambda$ with $f(A) \subseteq j_\alpha(X_\alpha)$ in the first place, in which case, as we have shown, this implies f factors through X_α via a continuous map, as desired. □

Proposition 2.9 (Hovey 2.4.5 & 2.4.6)**.** *The class of injective maps (resp. inclusions, closed T_1 inclusions) in* **Top** *is saturated.*

Proof. We know these three classes are closed under transfinite compositions (Corollary 2.5), so it suffices to show these classes are closed under pushouts and retracts. In what follows, fix a pushout diagram and a retract diagram of the following form:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"'] & C \arrow[d, "j"] \\
B \arrow[r, "g"'] \arrow[ur, phantom, "\lrcorner", very near end] & D
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, "f"'] \arrow[d, "j"'] \arrow[rr, bend left, equal] & B \arrow[r, "g"'] \arrow[d, "i"] & A \arrow[d, "j"] \\
C \arrow[r, "h"] \arrow[rr, bend right, equal] & D \arrow[r, "k"] & C
\end{tikzcd}$$

(i) First, consider the pushout diagram, and suppose i is injective. We wish to show j is injective. Suppose for the sake of a contradiction there existed distinct points $c_1, c_2 \in C$ such that $j(c_1) = j(c_2)$. Define $h : C \to \{1, 2, 3\}$ mapping $c_1 \mapsto 1$, $c_2 \mapsto 2$, and every other point in C maps to 3. Define $k : B \to \{1, 2, 3\}$ to map every point in $i(f^{-1}(c_1))$

to 1, every point in $i(f^{-1}(c_2))$ to 2, and every other point to 3. We may give $\{1,2,3\}$ the indiscrete topology so that h and k are continuous. Then clearly $h \circ f = k \circ i$, so there exists a map $\ell : D \to \{1,2,3\}$ such that $\ell \circ j = h$ and $\ell \circ g = k$. But we reach a contradiction, as $\ell(j(c_1)) = \ell(j(c_2))$, but $h(c_1) \neq h(c_2)$.

Now, consider the retract diagram, and suppose i is injective. We wish to show j is injective. Suppose $a_1, a_2 \in A$ such that $j(a_1) = j(a_2)$. Then $h(j(a_1)) = h(j(a_2))$, and $h \circ j = i \circ f$, so $i(f(a_1)) = i(f(a_2))$. Note that since $g \circ f = \mathrm{id}_A$, necessarily f is injective, and we are assuming i is injective, so $i(f(a_1)) = i(f(a_2)) \implies a_1 = a_2$.

(ii) First, consider the pushout diagram, and suppose i is an inclusion. We wish to show j is an inclusion. We know j is injective by (i). It remains to show that given $U \subseteq C$ open, there exists $V \subseteq D$ open with $j^{-1}(V) = U$. Given $U \subseteq C$ open, $f^{-1}(U)$ is open in A as f is continuous, and i is an inclusion, so there exists $W \subseteq B$ open with $i^{-1}(W) = f^{-1}(U)$. Now let $S = \{0,1\}$ be the Sierpinski space with open sets $\{\emptyset, \{1\}, \{0,1\}\}$. Define $h : C \to S$ to map every point of U to 1, and every other point to 0. Define $k : B \to S$ to map every point of W to 1, and every other point to 0. Clearly h and k are continuous. We claim $h \circ f = k \circ i$. Indeed, let $a \in A$. If $a \in f^{-1}(U)$, then $h(f(a)) = 1$, as $f(a) \in U$, while $k(i(a)) = 1$, as $a \in U = i^{-1}(W)$, so $i(a) \in W$. Conversely, if $a \notin f^{-1}(U)$, then $h(f(a)) = 0$ as $f(a) \notin U$, while $k(i(a)) = 0$, as $a \notin i^{-1}(W)$ meaning $i(a) \notin W$. Hence, there exists a (unique) continuous map $\ell : D \to S$ with $\ell \circ j = h$ and $\ell \circ g = k$. Define $V := \ell^{-1}(1)$, which is open in D as $\{1\}$ is open in S. Then finally, we claim $j^{-1}(V) = U$. Indeed,

$$j^{-1}(V) = j^{-1}(\ell^{-1}(1)) = (\ell \circ j)^{-1}(1) = h^{-1}(1) = U.$$

Thus j is an inclusion, as desired.

Now, consider the retract diagram, and suppose i is an inclusion. We wish to show j is an inclusion. We know j is injective by (i). It remains to show that given $U \subseteq A$ open, there exists $V \subseteq C$ open with $j^{-1}(V) = U$. Then since g is continuous, $g^{-1}(U)$ is open in B, and i is an inclusion, so there exists $W \subseteq D$ open with $i^{-1}(W) = g^{-1}(U)$. Then

$$j^{-1}(h^{-1}(W)) = (h \circ j)^{-1}(W) = (i \circ f)^{-1}(W) = f^{-1}(i^{-1}(W)) = f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U) = U,$$

and $h^{-1}(W)$ is open as h is continuous and W is open, so we are done, as if we set $V := h^{-1}(W)$, then we have shown V is open in C and $j^{-1}(V) = U$, as desired.

(iii) First, consider the pushout diagram, and suppose i is a closed T_1 inclusion. We wish to show j is a closed T_1 inclusion. We know j is an inclusion by (ii). It remains to show j is closed, and every point of $D \setminus j(C)$ is closed in D. First, we show closedness. Let $V \subseteq C$ be a closed set, we want to show $j(C)$ is closed in D. By definition of the colimit topology on D (which is the final topology on D induced by j and g by the discussion at the beginning of this chapter), in order to show $j(V)$ is closed in D it suffices to show that $j^{-1}(j(V))$ is closed in C and $g^{-1}(j(V))$ is closed in B. Since j is injective by (i), $j^{-1}(j(V)) = V$, which we have defined to be closed in A. Now, to show $g^{-1}(j(V))$ is closed, since i is a closed map and f is continuous, it suffices to show $i(f^{-1}(V)) = g^{-1}(j(V))$. First of all, let $b \in i(f^{-1}(V))$, then $b = i(a)$ for some $a \in A$ with $f(a) \in V$. Then $g(b) = g(i(a)) = j(f(a)) \in j(V)$, so $b \in g^{-1}(j(V))$. Conversely, let $b \in g^{-1}(j(V))$, so $g(b) = j(c)$ for some $c \in V$. Then by the explicit description of the colimit in **Set** and the fact that the forgetful functor **Top** → **Set** preserves colimits, there exists $a \in A$ such that $f(a) = c$ and $i(a) = b$. Then in particular, $f(a) = c \in V$, so $a \in f^{-1}(V)$, so $b \in i(f^{-1}(V))$ as desired. Hence, j is indeed a closed map. It remains to show that for all $d \in D \setminus j(C)$, d is a closed point in D. Given $d \in D \setminus j(C)$, by the explicit characterization of the colimit topology, it suffices to show that $j^{-1}(d)$ and $g^{-1}(d)$ are closed in C and B, respectively. First of all, $j^{-1}(d) = \emptyset$, which is closed. Now, since d is not in the image of j, d is not in the image of $g \circ i = j \circ f$, so $g^{-1}(d) \subseteq B \setminus i(A)$.

Thus since i is a closed T_1 inclusion, in order to show $g^{-1}(d)$ is closed, it suffices to show that $g^{-1}(d)$ is a singleton. Suppose for the sake of a contradiction there existed distinct points $x, y \in B$ with $g(x) = g(y) = d$. Define $h : C \to \{1, 2, 3\}$ to map every point to 3. Define $k : B \to \{1, 2, 3\}$ to map $x \mapsto 1$, $y \mapsto 2$, and every other point in B maps to 3. Endow $\{1, 2, 3\}$ with the indiscrete topology so that h and k are continuous. Note $h \circ f = k \circ i$: given $a \in A$, since $x, y \in g^{-1}(d) \subseteq B \setminus i(A)$, $i(a)$ does not equal x or y, thus $k(i(a)) = 3 = h(f(a))$. Hence by the definition of the colimit, there exists a map $\ell : D \to X$ such that $\ell \circ g = k$ and $\ell \circ j = h$. Then we reach a contradiction, as $k(x) \neq k(y)$ but $\ell(g(x)) = \ell(g(y)) = \ell(d)$. Hence, $g^{-1}(d) \notin i(A)$ must have been a singleton in the first place, meaning $g^{-1}(d)$ is closed in B as desired.

Now, consider the retract diagram, and suppose i is a closed T_1 inclusion. We wish to show j is a closed T_1 inclusion. We know j is an inclusion by (ii). It remains to show j is closed, and every point of $C \setminus j(A)$ is closed in C. First, we show closedness. Let $V \subseteq A$ be closed. First, we claim $j(V) = h^{-1}(i(g^{-1}(V)))$. Given $c \in j(V)$, $c = j(a)$ for some $a \in V$, in which we have $h(c) = h(j(a)) = i(f(a))$, and $g(f(a)) = a \in V$, so $f(a) \in g^{-1}(V)$, meaning $h(c) \in i(g^{-1}(V))$, so $c \in h^{-1}(i(g^{-1}(V)))$. Conversely, given $c \in h^{-1}(i(g^{-1}(V)))$, so $h(c) = i(b)$ for some $b \in B$ with $g(b) \in V$. Then

$$c = k(h(c)) = k(i(b)) = j(g(b)) \in j(V),$$

as desired. Thus, we have shown $j(V) = h^{-1}(i(g^{-1}(V)))$.Note that since $V \subseteq A$ is closed and g is continuous, $g^{-1}(V)$ is closed in B. Since i is a closed map, $i(g^{-1}(V))$ is closed in D. Finally, since h is continuous, $h^{-1}(i(g^{-1}(V))) = j(V)$ is closed in C, as desired. It remains to show that for all $c \in C \setminus j(A)$ that c is a closed point in C. Given $c \in C \setminus j(A)$, note that $h(c) \notin i(B)$, as if $h(c) = i(b)$ for some $b \in B$, then we would have $c = k(h(c)) = k(i(b)) = j(g(b))$, yet we chose c not in the image of j. Hence, since i is a closed T_1 inclusion, and $h(c)$ is not in the image of i, $h(c)$ is a closed point in D. Then note that since $k \circ h = \mathrm{id}_C$, h is injective, so $c = h^{-1}(h(c))$ is a closed point, as it is the preimage of the closed set $\{h(c)\}$ along the continuous map h. □

Lemma 2.10 (Hovey 2.4.8)**.** *$\mathcal{W} \cap \mathcal{T}$ is closed under transfinite compositions, where $\mathcal{T}$ denotes the class of closed T_1 inclusions.*

Proof. Let λ be an ordinal, and let $X : \lambda \to \mathbf{Top}$ be a λ-sequence such that for all $\beta + 1 < \lambda$, the map $X_\beta \to X_{\beta+1}$ belongs to $\mathcal{W} \cap \mathcal{T}$. Let $j : X \to \underline{X_\lambda}$ be a colimit cone for X. By Corollary 2.5, we know that $j_0 : X \to X_\lambda$ is a closed T_1 inclusion, so it remains to show that $\pi_n(j_0, x_0) : \pi_n(X_0, x_0) \to \pi_n(X_\lambda, j_0(x_0))$ is an isomorphism for all $n \geq 0$ and $x_0 \in X_0$.

First we show surjectivity. Suppose we are given $x_0 \in X_0$ and a continuous map $f : (S^n, *) \to (X_\lambda, j_0(x_0))$. Since S^n is compact, by □

Oops, I left this unfinished. Honestly I don't really want to type out this argument, leaving for now.

Proposition 2.11. *The domains of I' (resp. J) are small relative to I'-*cell.

Proof. By Lemma 2.7, every space is small relative to the inclusions, and in particular every space is small relative to the class $\mathcal{T}$ of closed T_1 inclusions. Hence, it suffices to show that J-cell, I'-cell $\subseteq \mathcal{T}$. We showed above in Proposition 2.9 that $\mathcal{T}$ is closed under transfinite composition and pushouts, and clearly every map in I' and J is a closed T_1 inclusion, so the desired result follows. □

Lemma 2.12 (Hovey Lemma 2.4.4)**.** *The weak equivalences in* **Top** *are closed under retracts and satisfy 2-of-3 axiom (so that in particular the weak equivalences form a subcategory, as clearly identities are weak equivalences).*

Proof. First we show that weak equivalences satisfy 2-of-3. Let $f : X \to Y$ and $g : Y \to Z$ be continuous functions of topological spaces.

First of all, suppose f and g are both weak equivalences. Then by functoriality of π_n, since $\pi_n(f,x)$ and $\pi_n(g,f(x))$ are isomorphisms for all $x \in X$, $\pi_n(g \circ f, x) = \pi_n(g, f(x)) \circ \pi_n(f,x)$ is likewise an isomorphism for all $x \in X$, so that $g \circ f$ is a weak equivalence.

Now, suppose that $g \circ f$ and g are weak equivalences. Pick a point $x \in X$. We wish to show that $\pi_n(f,x) : \pi_n(X,x) \to \pi_n(Y, f(x))$ is an isomorphism for all $n \geq 0$. We know that $\pi_n(g \circ f, x)$ is an isomorphism, and $\pi_n(g, f(x))$ is an isomorphism, say with inverse, φ, so that

$$\varphi \circ \pi_n(g \circ f, x) = \varphi \circ \pi_n(g, f(x)) \circ \pi_n(f,x) = \pi_n(f,x)$$

is an isomorphism, as it is a composition of isomorphisms.

Now, suppose that $g \circ f$ and f are weak equivalences. Pick a point $y \in Y$. Since $\pi_0(f)$ is an isomorphism, there exists a point $x \in X$ such that $f(x)$ belongs to the path component containing y, so that there exists some $\alpha : I \to Y$ with $\alpha(0) = f(x)$ and $\alpha(1) = f(y)$. Then consider the following diagram

$$\begin{tikzcd}
\pi_n(Y,y) \arrow[r, "\pi_n(g,y)"] \arrow[d] & \pi_n(Z, g(y)) \arrow[d] \\
\pi_n(Y, f(x)) \arrow[r, "\pi_n(g,f(x))"] & \pi_n(Z, g(f(x)))
\end{tikzcd}$$

where the left arrow is the isomorphism given by conjugation by the path α, and the right arrow is the isomorphism given by conjugation by the path $g \circ \alpha$. It is tedious yet straightforward to verify that the diagram commutes. Furthermore, we know that $\pi_n(f,x)$ and $\pi_n(g \circ f, x) = \pi_n(g, f(x)) \circ \pi_n(f,x)$ are isomorphisms for all n, so that if we denote the inverse of $\pi_n(f,x)$ by φ, then

$$\pi_n(g \circ f, x) \circ \varphi = \pi_n(g, f(x)) \circ \pi_n(f,x) \circ \varphi = \pi_n(g, f(x))$$

is an isomorphism, as it is given as a composition of isomorphisms. Hence, the top arrow must likewise be an isomorphism, precisely the desired result.

The fact that weak equivalences in **Top** are closed under retracts is entirely straightforward and follows from the fact that the functors π_n preserve retract diagrams and that the class of isomorphisms in any category is closed under retracts. $\square$

Proposition 2.13 (Hovey 2.4.9)**.** $_\perp(J_\perp) \subseteq \mathcal{W} \cap {}_\perp(I'_\perp)$.

Proof. First, in order to show $_\perp(J_\perp) \subseteq {}_\perp(I'_\perp)$, It suffices to show that $J \subseteq I'$-cell, as by Corollary 1.16 we would have $J \subseteq {}_\perp(I'_\perp)$, and

$$J \subseteq {}_\perp(I'_\perp) \implies {}_\perp(J_\perp) \subseteq {}_\perp(({}_\perp(I'_\perp))_\perp) = {}_\perp(I'_\perp),$$

where the implication and equality both follow from Lemma 1.11 which gives that

$$A \subseteq B \implies {}_\perp(A_\perp) \subseteq {}_\perp(B_\perp) \quad \text{and} \quad ({}_\perp(A_\perp))_\perp = A_\perp.$$

Now, to show $J \subseteq I'$-cell, first consider the composition $j_n : D^n \hookrightarrow S^n \hookrightarrow D^{n+1}$, where the first map is the pushout

$$\begin{tikzcd}
S^{n-1} \arrow[r, hook] \arrow[d, hook] & D^n \arrow[d] \\
D^n \arrow[r] \arrow[ur, phantom, "\lrcorner", very near start] & S^n
\end{tikzcd}$$

obtained by gluing two copies of D^n along their boundary, and the second map map is simply the inclusion $S^n \hookrightarrow D^{n+1}$, which can be written as the pushout

$$\begin{tikzcd}
S^n \arrow[r, equal] \arrow[d, hook] & S^n \arrow[d, hook] \\
D^{n+1} \arrow[r, equal] \arrow[ur, phantom, "\lrcorner", very near start] & D^{n+1}
\end{tikzcd}$$

It can be seen that j_n includes D^n as a hemisphere of $S^n = \partial D^{n+1} \subseteq D^{n+1}$. Note that $D^n \times I$ is homeomorphic to D^{n+1} ("smooth out" the sharp edges of the cylinder) via some homeomorphism $h_n : D^{n+1} \to D^n \times I$, and in particular, we may define h_n so that $h_n(j_n(D^n)) = D^n \times \{0\} \subseteq D^n \times I$ by squashing the hemisphere $j_n(D^n)$ to be one of the faces of the cylinder $D^n \times I$, in which case $h_n \circ j_n : D^n \to D^n \times I$ is precisely the inclusion $D^n \hookrightarrow D^n \times I$ sending $x \mapsto (x, 0)$, and since $j_n \in I'$-cell, $h_n \circ j_n \in I'$-cell by Lemma 1.13.

Now, we claim that ${}_\perp(J_\perp) \subseteq \mathcal{W}$. First note that by Corollary 1.18 and Proposition 2.11, every map in ${}_\perp(J_\perp)$ is a retract of an element of J-cell. Furthermore, we know that $\mathcal{W}$ is closed under retracts (Lemma 2.12), so that it suffices to show that J-cell $\subseteq \mathcal{W}$. We claim it suffices to show that pushouts of maps in J are weak equivalences. Supposing we had shown this, we would have that pushouts of maps in J are weak equivalences and T_1 inclusions, as $J \subseteq \mathcal{T}$ and $\mathcal{T}$ is saturated by Proposition 2.9. Then by Lemma 2.10, we would have that J-cell $\subseteq \mathcal{W} \cap \mathcal{T}$, precisely the desired result.

Now, let $\mathcal{S}$ be the class of *inclusions of a deformation retract*, i.e., those **injective** maps $i : A \to B$ such that there exists a homotopy $H : B \times I \to B$ with $H(i(a), t) = i(a)$ for all $a \in A$, $H(b, 0) = b$ for all $b \in B$, and $H(b, 1) = i(r(b))$ for all $b \in B$ for some map $r : B \to A$.[3] We will show the following:

(1) $\mathcal{S} \subseteq \mathcal{W}$.

It suffices to show that if $i : A \to B$ belongs to $\mathcal{S}$, then i is a homotopy equivalence. Indeed, given $i : A \to B$, let $H : B \times I \to B$ and $r : B \to A$ be a homotopy and retract satisfying the conditions above. Then in particular, H is a homotopy between id_B (at time $t = 0$) and $i \circ r$ (at time $t = 1$). It remains to show that $r \circ i = \mathrm{id}_A$. First of all, note that since $H(b, 1) = i(r(b))$ for all $b \in B$, we have $H(i(a), 1) = i(r(i(a)))$. Yet, we also know that $H(i(a), t) = i(a)$ for all $t \in I$, so $i(r(i(a))) = i(a)$, and i is injective so $r(i(a)) = a$.

(2) $J \subseteq \mathcal{S}$.

For $n \geq 0$, let $j_n : D^n \hookrightarrow D^n \times I$ denote the inclusion of D^n as the subset $D^n \times \{0\}$. Define a deformation retract $H : D^n \times I \times I \to D^n \times I$ by $(x, s, t) \mapsto (x, s(1 - t))$. Then indeed we have $H(j_n(x), t) = H(x, 0, t) = (x, 0) = j_n(x)$ for all $x \in D^n$, $H(x, t, 0) = (x, t(1-0)) = (x, t)$ for all $(x, t) \in D^n \times I$, and $H(x, t, 1) = (x, t(1-1)) = (x, 0) = j_n(r(x))$ for all $(x, t) \in D^n \times I$, where $r : D^n \times I \to D^n$ is the projection onto time zero sending $(x, t) \mapsto (x, 0)$. Finally, j_n is clearly injective. Thus, indeed $J \subseteq \mathcal{S}$.

(3) $\mathcal{S}$ is closed under pushouts.

Suppose we are given a pushout diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"'] & C \arrow[d, "j"] \\
B \arrow[r, "g"'] & D \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}$$

where $i \in \mathcal{S}$. Then we wish to show j in $\mathcal{S}$. First, we know j is injective by Proposition 2.9. Now, we look to construct H and r. Let $K : B \times I \to B$ and $r' : B \to A$ be maps satisfying the conditions for i to be an inclusion of a deformation retract.

We wish to define a homotopy $H : D \times I \to D$. Then I is a locally compact Hausdorff space (in particular, it is compact and Hausdorff), so that the functor $- \times I : \mathbf{Top} \to \mathbf{Top}$

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[3] Hovey has a typo here, namely, he does not specify that i must be injective. Without this specification, his assertion fails. For example, take $A = \mathbb{R}^2$, $B = \mathbb{R}$, $i(x, y) = x$, $H(b, t) = b$, and $r(b) = (b, 0)$. Then i is an inclusion of a deformation retract according to Hovey's "definition," but i is not injective and r is not a retract.

preserves colimits (Proposition 2.1), meaning the following is a pushout diagram:

$$\begin{tikzcd}
A \times I \arrow[r, "f \times \mathrm{id}_I"] \arrow[d, "i \times \mathrm{id}_I"'] & C \times I \arrow[d, "j \times \mathrm{id}_I"] \\
B \times I \arrow[r, "g \times \mathrm{id}_I"'] & D \times I \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Then by the universal property of the pushout, there is a map $H : D \times I \to D$ (the dashed line) such that the following diagram commutes

$$\begin{tikzcd}
A \times I \arrow[r, "f \times \mathrm{id}_I"] \arrow[d, "i \times \mathrm{id}_I"'] & C \times I \arrow[d, "j \times \mathrm{id}_I"] \arrow[dr, bend left, "\pi_1"] & \\
B \times I \arrow[r, "g \times \mathrm{id}_I"'] \arrow[dr, bend right, "K"'] & D \times I \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "H"] & C \arrow[d, "j"] \\
& B \arrow[r, "g"'] & D
\end{tikzcd}$$

Now, note $r' \circ i = \mathrm{id}_A$. Indeed, given $a \in A$, we have $i(r'(i(a))) = K(i(a), t) = i(a)$ and i is injective, so that $r'(i(a)) = a$, as desired. Hence, there exists a unique map $r : D \to C$ (the dashed line) such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"'] & C \arrow[d, "j"] \arrow[dr, bend left, equal] & \\
B \arrow[r, "g"'] \arrow[dr, bend right, "r'"'] & D \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "r"] & \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

Now we claim that our constructions H and r endow j with the structure of an inclusion of a deformation retract, as desired. First $c \in C$, we wish to show $H(j(c), t) = j(c)$ for all t. Indeed, we have

$$H(j(c), t) = H(j \times \mathrm{id}_I(c, t)) = j(\pi_1(c, t)) = j(c).$$

Given $d \in D$, we want to show $H(d, 0) = d$. By the explicit description of the colimit in **Top**, we know that every element of D is in the image of either j or g. If $d = j(c)$ for some c, then we have just shown $H(d, 0) = H(j(c), 0) = j(c) = d$, as desired. On the other hand, if $d = g(b)$ for some $b \in B$ we have

$$H(d, 0) = H(g \times \mathrm{id}_I(b, 0)) = g(K(b, 0)) = g(b) = d.$$

Finally, we claim that $H(d, 1) = j(r(d))$ for all $d \in D$. If $d = j(c)$ for some $c \in C$, then we have

$$H(d, 1) = H(j(c), 1) = j(c) = j(r(j(c))) = j(r(d)),$$

as desired. On the other hand, if $d = g(b)$ for some $b \in B$, then

$$H(d, 1) = H(g \times \mathrm{id}_I(b, 1)) = g(K(b, 1)) = g(i(r'(b))) = j(f(r'(b))) = j(r(g(b))) = j(r(d)). \qquad \square$$

Proposition 2.14 (Hovey 2.4.10). $I'_\perp \subseteq \mathcal{W} \cap J_\perp$

Proof. First, by Proposition 2.13 we know ${}_\perp(J_\perp) \subseteq {}_\perp(I'_\perp)$, and this implies $I'_\perp \subseteq J_\perp$, as by Lemma 1.11 we have

$${}_\perp(J_\perp) \subseteq {}_\perp(I'_\perp) \implies J_\perp = ({}_\perp(J_\perp))_\perp \supseteq ({}_\perp(I'_\perp))_\perp = I'_\perp.$$

Thus, it suffices to show that $I'_\perp \subseteq \mathcal{W}$. Now, suppose $p : (X, x_0) \to (Y, p(x_0))$ is in $I'_\perp$. We wish to show that the map $\pi_n(p, x_0) : \pi_n(X, x_0) \to \pi_n(Y, p(x_0))$ is an isomorphism for all n.

First we show that $\pi_n(p, x_0)$ is surjective. Let $g : (S^n, *) \to (Y, p(x_0))$ be a map. Then we have the following commutative diagram

$$\begin{tikzcd} * \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ S^n \arrow[r, "g"] & Y \end{tikzcd}$$

where the top arrow picks out x_0. Note that the map $* \to S^n$ may be realized as a pushout of the diagram $D^n \leftarrow S^{n-1} \to *$, so that $* \to S^n$ belongs to I'-cell, and therefore ${}_\perp(I'_\perp)$ by Corollary 1.16, and $p \in I'_\perp$, so $* \to S^n$ has the left lifting property against p. Thus, the above diagram has a lift $f : (S^n, *) \to (X, x_0)$ such that $p \circ f = g$, so that $\pi_n(p, x_0)([f]) = [p \circ f] = [g]$, as desired.

Finally, we show that $\pi_n(p, x_0)$ is injective. Suppose we have two maps $f, g : (S^n, *) \to (X, x_0)$ such that $p \circ f$ and $p \circ g$ represent the same element of $\pi_n(Y, p(x_0))$. Then there is a homotopy $H : S^n \times I \to Y$ such that for all $s \in S^n$ and $t \in I$, $H(s, 0) = p(f(s))$, $H(s, 1) = p(g(s))$, and $H(*, t) = p(x_0)$. By the universal property of the quotient, H induces a map $\overline{H} : S^n \wedge I_+ := (S^n \times I)/(* \times I)$ sending the equivalence class $[s, t] \mapsto H(s, t)$. Hence, the following diagram commutes:

$$\begin{tikzcd} S^n \vee S^n \arrow[r, "f \vee g"] \arrow[d, hook] & X \arrow[d, "p"'] \\ S^n \wedge I_+ \arrow[r, "\overline{H}"] & Y \end{tikzcd}$$

where the left arrow is an element of I'-cell, as it may be obtained by attaching an $n + 1$ cell to $S^n \vee S^n$ (when $n = 0$, the attachning map is obvious; when $n > 0$, the attaching map is the quotient map $S^n \twoheadrightarrow S^n \vee S^n$ obtained by collapsing the equator). Thus, by similar reasoning to above there exists a lift $\overline{K} : S^n \wedge I_+ \to X$. Then if we define K to be the composition $S^n \times I \twoheadrightarrow S^n \wedge I_+ \xrightarrow{\overline{K}} X$, this gives us the desired homotopy between f and g: given $s \in S^n$ and $t \in I$, we have $K(s, 0) = \overline{K}([s, 0]) = f(s)$, $K(s, 1) = \overline{K}([s, 1]) = g(s)$, and $K(*, t) = \overline{K}([*, t]) = \overline{K}([*, 0]) = (*) = x_0$. ☐

In what follows, given continuous maps $p : X \to Y$ and $i : A \to B$, let $Q(i, p) : X^B \to P(i, p) := X^A \times_{Y^A} Y^B$ denote the map obtained by the universal property of the fiber product (pullback) via the maps $i^* \circ p_* : X^B \to Y^A$ and $p_* \circ i^* : X^B \to Y^A$.

Lemma 2.15 (Hovey 2.4.11)**.** *Suppose $p : X \to Y$ is a map. Then $p \in I'_\perp$ if and only if the map $Q(i, p) : X^B \to P(i, p) := X^A \times_{Y^A} Y^B$ is surjective for all maps $i : A \to B$ in I'. In particular, if $Q(i, p) \in \mathcal{W} \cap J_\perp$ for all $i \in I'$, then $p \in I'_\perp$.*

Proof. First of all, suppose $p \in I'_\perp$, and let $i : A \to B$ in I'. We wish to show $Q(i, p)$ is surjective. We know the forgetful functor **Top** $\to$ **Set** preserves limits, so by the explicit description of limits in **Set** and Remark 1.7, the set $P(i, p)$ is given by pairs of maps $(A \xrightarrow{\alpha} X, B \xrightarrow{\beta} Y)$ such that $p \circ \alpha = \beta \circ i$, and the map $Q(i, p) : X^B \to P(i, p)$ sends an arrow $\ell : B \to X$ to the pair $(\ell \circ i, p \circ \ell)$. Now to see $Q(i, p)$ is surjective, suppose there exists $(\alpha, \beta) \in P(i, p)$ so that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "\beta"'] & Y \end{tikzcd}$$

Then because $p \in I'_{\perp}$ and $i \in I'$, there exists a lift $\ell : B \to X$, in which case $Q(i,p)(\ell) = (\ell \circ i, p \circ \ell) = (\alpha, \beta)$, as desired.

Conversely, suppose that we are given a map $p : X \to Y$ such that $Q(i,p)$ is surjective for all maps $i : A \to B$ in I'. Then we wish to show that $p \in I'_{\perp}$. Let $i : A \to B$ belong to I', and suppose we are given a lifting problem of the same form as the above diagram. Then since $p \circ \alpha = \beta \circ i$, the pair (α, β) is an element of $P(i,p)$. Since $Q(i,p)$ is surjective, there exists an arrow $\ell : B \to X$ such that $Q(i,p)(\ell) = (\alpha, \beta)$. But $Q(i,p)(\ell) = (\ell \circ i, p \circ \ell)$, so $\ell \circ i = \alpha$ and $p \circ \ell = \beta$, meaning ℓ is a lift, as desired. Thus indeed $p \in I'_{\perp}$.

Thus, in order to show the second part, it suffices to show that any map belonging to $\mathcal{W} \cap J_{\perp}$ is surjective. To see this, suppose $q : W \to Z$ belongs to $\mathcal{W} \cap J_{\perp}$, and let $z \in Z$. Pick any point $w \in W$, and consider the map $\beta : S^0 = \{0,1\} \to Z$ sending $0 \mapsto q(w)$ and $1 \mapsto z$. Then since $q \in \mathcal{W}$, the homomorphism $\pi_0(q,w) : (W,w) \to (Z, q(w))$ is surjective, so there exists a map $\alpha : (S^0, 0) \to (W, w)$ and a homotopy $H : S^0 \times I \to Z$ such that $H(s,0) = q(\alpha(s))$, $H(s,1) = \beta(s)$, and $H(0,t) = q(w)$ for all $s \in S^0$ and $t \in I$. Then we have a lifting problem of the following form

$$\begin{tikzcd}
* \arrow[r, "\alpha(1)"] \arrow[d, "0"'] & W \arrow[d, "q"] \\
I \arrow[r, "{H(1,-)}"'] & Z
\end{tikzcd}$$

$H(1,-)$ is continuous as it is $H^{\flat}(1)$, where $H^{\flat} : S^0 \to W^I$ is the adjoint of $H : S^0 \times I \to W$ (see the proof of Proposition 2.1 and recall how to construct the adjoint of a morphism given the unit and counit of the adjunction). Then we may identify the left arrow with the map $D^0 \to D^0 \times I$, in which case since $q \in J_{\perp}$ there exists a lift $\ell : I \to W$ such that $q \circ \ell = H(1,-)$. In particular, $z = \beta(1) = H(1,1) = q(\ell(1))$, so that z does belong to the image of q, as desired. □

Lemma 2.16 (Hovey 2.4.13)**.** *Suppose $p : X \to Y$ belongs to $J_{\perp}$ and $i : S^{n-1} \hookrightarrow D^n$ belongs to I'. Then the map $Q(i,p)$ belongs to $J_{\perp}$.*

Proof. Suppose we are given a map $j : D^m \times \{0\} \hookrightarrow D^m \times I$ belonging to J and a lifting problem of the following form

$$\begin{tikzcd}
D^m \arrow[r] \arrow[d, hook, "j"'] & X^{D^n} \arrow[d, "{Q(i,p)}"'] \\
D^m \times I \arrow[r] & P(i,p) = X^{S^{n-1}} \times_{Y^{S^{n-1}}} Y^{D^n}
\end{tikzcd}$$

In order to show that any such diagram has a lift, by Proposition 2.1 and Proposition 1.22, since S^{n-1} and D^n are locally compact Hausdorff, it suffices to show that any lifting problem of the following form has a solution:

$$\begin{tikzcd}
D^m \times \{0\} \times D^n \coprod_{D^m \times \{0\} \times S^{n-1}} D^m \times I \times S^{n-1} \arrow[r] \arrow[d, "{(i \times \mathrm{id}_{D^n}, \mathrm{id}_{D^m \times I} \times i)}"'] & X \arrow[d, "p"] \\
D^m \times I \times D^n \arrow[r] & Y
\end{tikzcd}$$

For ease of notation, write f for the map on the left. Since $p \in J_{\perp}$, in order to show that the diagram has a lift, it suffices to show that $f \in {}_{\perp}(J_{\perp})$. Note that since ${}_{\perp}(J_{\perp})$ is characterized by a lifting property, it is closed under composition with isomorphisms (homeomorphisms). Furthermore, since $J \subseteq {}_{\perp}(J_{\perp})$, it suffices to show that f is homeomorphic to the second map in the factorization $D^{m+n} \cong D^{m+n} \times \{0\} \hookrightarrow D^{m+n} \times I \in J$. Note that $f = \mathrm{id}_{D^m} \times g$, where

$$g : \{0\} \times D^n \coprod_{\{0\} \times S^{n-1}} I \times S^{n-1} = (\{0\} \times D^n) \cup (I \times S^{n-1}) \hookrightarrow I \times D^n.$$

The space $(\{0\} \times D^n) \cup (I \times S^{n-1})$ may be obtained by removing one of the ends of the hollow cylinder $\partial(I \times D^n)$. By flattening the edges of this cylinder-with-a-missing-end, we get a disk homeomorphic to D^n. Thus g is homeomorphic to the map $\{0\} \times D^n \hookrightarrow I \times D^n$, so that $f = \mathrm{id}_{D^m} \times g$ is homeomorphic to the inclusion $D^{m+n} \times \{0\} \hookrightarrow D^{m+n} \times I$ as desired. □

Corollary 2.17 (Hovey 2.4.14). *Every topological space is fibrant, i.e., given a space X, the unique map $X \to *$ is an element of $J_\perp$. In particular, the map $X^{D^n} \to X^{S^{n-1}}$ belongs to $J_\perp$ for all $n \geq 0$.*

Proof. Suppose we are given a space X, a map $j : D^n \hookrightarrow D^n \times I$ belonging to J, and a lifting problem of the following form

$$\begin{tikzcd}
D^n \arrow[r, "f"] \arrow[d, "j"'] & X \arrow[d] \\
D^n \times I \arrow[r] & *
\end{tikzcd}$$

Then it is straightforward to see that the composition $f \circ \pi_1$, where $\pi_1 : D^n \times I \to D^n$ is the canonical projection, is a lift for the diagram. Hence, X is indeed fibrant as desired. Now, suppose we are given a map $i : S^{n-1} \hookrightarrow D^n$ belonging to I'. It is straightforward to check that the following is a pullback diagram:

$$\begin{tikzcd}
X^{S^{n-1}} \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & *^{D^n} \arrow[d] \\
X^{S^{n-1}} \arrow[r] & *^{S^{n-1}}
\end{tikzcd}$$

where the top arrow sends a map $f : S^{n-1} \to X$ to unique arrow $D^n \to *$ (it is continuous as $*^{D^n}$ is a singleton). Hence, by Lemma 2.16 the map $X^{D^n} \to P(i, p) = X^{S^{n-1}}$ sending $f : D^n \to X$ to the composition $S^{n-1} \to D^n \xrightarrow{f} X$ is fibrant (belongs to $J_\perp$), as desired. □

Lemma 2.18 (Hovey 2.4.15). *If $p : X \to Y$ is a weak equivalence (belongs to $\mathcal{W}$), then $p^{D^n} : X^{D^n} \to Y^{D^n}$ is also a weak equivalence.*

Proof. We claim it suffices to show that the map $j_Z : Z \to Z^{D^n}$ that takes z to the constant map at z is a homotopy equivalence for all spaces Z. Indeed, supposing this had been shown, then consider the following diagram

$$\begin{tikzcd}
X \arrow[r, "p"] \arrow[d, "j_X"'] & Y \arrow[d, "j_Y"] \\
X^{D^n} \arrow[r, "p_*"'] & Y^{D^n}
\end{tikzcd}$$

First of all, it is straightforward to see that this diagram commutes. By 2-of-3 (Lemma 2.12) we know that $p_* \circ j_X = j_Y \circ p$ is a weak equivalence, if j_Y is, as p is a weak equivalence. Furthermore, if j_X is a weak equivalence, then by 2-of-3 again, it would have to hold that p_* is a weak equivalence, as desired.

Now, we claim that j_Z is homotopy equivalence. First of all, it is well-defined, as given $z \in Z$, $j_Z(z) : D^n \to Z$ is a constant map, which is always continuous, so that $j_Z(z)$ is indeed an element of Z^{D^n}. Now, to see that j_Z is continuous, recall the topology on Z^{D^n} is that with subbasis given by sets of the form

$$S(K, U) := \{f \in \mathbf{Top}(D^n, Z) : f(K) \subseteq U\}$$

for $K \subseteq D^n$ compact and $U \subseteq Z$ open. Hence in order to show j_Z is continuous, it suffices to show $j_Z^{-1}(S(K, U))$ is open in Z for all $K \subseteq D^n$ compact and $U \subseteq Z$ open. Indeed, we have

$$j_Z^{-1}(S(K, U)) = \{z \in Z : j_Z(z)(K) \subseteq U\} = \{z \in Z : z \in U\} = U,$$

which is open as desired. Hence, j_Z is well-defined and continuous. Now, it remains to show that it is a homotopy equivalence. Define the map $q_Z : Z^{D^n} \to Z$ to be evaluation at 0 (where by 0 we mean the origin in D^n), so that $q_Z(f : D^n \to Z) := f(0)$. To see this is continuous, note that given $U \subseteq Z$ open, that

$$q_Z^{-1}(U) := \{f \in \mathbf{Top}(D^n, Z) : f(0) \in U\} = S(\{0\}, U),$$

which is open in Z^{D^n}, as desired. Now, the composite $q_Z \circ j_Z$ is the identity, and the composite $j_Z \circ q_Z$ is homotopic to the identity by the homotopy $H : Z^{D^n} \times I \to Z^{D^n}$ defined by $H(f, t)(x) = f(tx)$. It remains to show H is continuous. To do so, it suffices to show that $H^\sharp : Z^{D^n} \times I \times D^n \to Z$ mapping $(f, t, x) \mapsto f(tx)$ is continuous, as D^n is locally compact Hausdorff and H is the adjoint of $H^\sharp$ (Proposition 2.1). Note that $H^\sharp$ factors as

$$\begin{tikzcd} Z^{D^n} \times I \times D^n \arrow[r] & Z^{D^n} \times D^n \arrow[r] & Z \\ (f,t,x) \arrow[r] & (f,tx) \arrow[r] & f(tx), \end{tikzcd}$$

and this composition is continuous as the first arrow is the product of $\mathrm{id}_{Z^{D^n}}$ with the multiplication map $I \times D^n \to D^n$ sending $(t, x) \mapsto tx$, and the second arrow is simply the evaluation map, which is also continuous (see the proof of Proposition 2.1). □

Lemma 2.19 (Hovey 2.4.16)**.** *Suppose $p : X \to Y$ belongs to $J_\perp$, and $x \in X$. Let $F := p^{-1}(p(x))$, and $i : F \hookrightarrow X$ denote the inclusion. Then there is a long exact sequences*

$$\begin{tikzcd} & \cdots \arrow[r] & \pi_{n+1}(Y,p(x)) \arrow[dll, "d_n"'] \\ \pi_n(F,x) \arrow[r, "\pi_n(i,x)"'] & \pi_n(X,x) \arrow[r, "\pi_n(p,x)"] & \pi_n(Y,p(x)) \arrow[dll, "d_{n+1}"] \\ \pi_{n-1}(F,x) \arrow[r, "\pi_{n-1}(i,x)"'] & \cdots \arrow[r, "\pi_0(p,x)"'] & \pi_0(Y,p(x)) \end{tikzcd}$$

which is natural with respect to commutative squares

$$\begin{tikzcd} X \arrow[r] \arrow[d, "p"'] & X' \arrow[d, "p'"] \\ Y \arrow[r] & Y' \end{tikzcd}$$

where $p, p' \in J_\perp$. Here d_n is a group homomorphism $\pi_n(Y, p(x)) \to \pi_{n-1}(F, x)$ when $n > 1$.

Proof. First, fix $n \geq 0$. We define $d_{n+1} : \pi_{n+1}(Y, p(x)) \to \pi_n(F, x)$. To start with, fix a homeomorphism $k_n : D^{n+1} \cong D^n \times I$, and write $\partial k_n : \partial(D^{n+1}) = S^n \cong \partial(D^n \times I)$ for the restriction of k_n to $\partial(D^{n+1}) = S^n$. We may represent an element of $\pi_{n+1}(Y, p(x))$ by a map $f : (D^{n+1}, S^n) \to (Y, p(x))$. Then we may construct the following lifting problem

$$\begin{tikzcd} D^n \arrow[r, "d\mapsto x"] \arrow[d, hook, "d\mapsto(d,0)"'] & X \arrow[d, "p"] \\ D^n \times I \arrow[r, "f\circ k_n^{-1}"] & Y \end{tikzcd}$$

Since $p \in J_\perp$, this diagram has a lift $D^n \times I \to X$. Let ℓ denote the restriction of this lift to $\partial(D^n \times I)$. Note that the image of ℓ is contained in F, as given $z \in \partial(D^n \times I)$, we have $p(\ell(z)) = f(k_n^{-1}(z))$ Then define $d_{n+1}f := \ell|_{\partial(D^n\times I)} \circ \partial k_n$. First, note that $d_{n+1}f : S^n \to F$

□

Finish or add reference (Hatcher Theorem 4.41)

Lemma 2.20. *Let $f : X \to Y$ be a continuous map between compact Hausdorff spaces. Then given $A \subseteq X$ and $B \subseteq Y$, A and B are closed iff they are compact, in which case $f^{-1}(B)$ and $f(A)$ are both closed and compact*

Proof. nLab reference $\square$

Add proper reference or complete proof

Lemma 2.21. *Let Z be a topological space. Then for all $n \geq 0$ and for all points $* \in S^n$, consider the pullback square*

$$\begin{tikzcd}
S^{n-1} \arrow[r, "i"] \arrow[d, "h"'] & D^n \arrow[d, "j"] \\
* \arrow[r, "k"] & S^n \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd} \tag{6}$$

Explicitly, when $n > 0$, j is the quotient map which collapses the boundary of D^n to the point $*$ to obtain S^n. Then the following is a pullback diagram

$$\begin{tikzcd}
Z^{S^n} \arrow[r, "j^*"] \arrow[d, "k^*"'] & Z^{D^n} \arrow[d, "i^*"] \\
Z^* \arrow[r, "h^*"] & Z^{S^{n-1}}
\end{tikzcd} \tag{7}$$

where each arrow is the pullback of the corresponding arrow in diagram (6). In particular, for all $* \in S^n$, the evaluation map $Z^{S^n} \to Z$ sending $\alpha \mapsto \alpha(*)$ belongs to $J_\perp$.

Proof. Note if $n = 0$, diagram (7) is obviously a pullback diagram, as it simply becomes

$$\begin{tikzcd}
Z^{S^0} \cong Z \times Z \arrow[r] \arrow[d] & Z^* \cong Z \arrow[d] \\
Z^* \cong Z \arrow[r] & Z^{\emptyset} \cong *
\end{tikzcd}$$

(Check the arrows are the obvious ones under these isomorphisms). Now suppose $n > 0$ in diagram (7). First of all the diagram clearly commutes as (6) does: given $f \in Z^{S^n}$, $i^*(j^*(f)) = f \circ j \circ i = f \circ k \circ h = h^*(k^*(f))$. Now, given arrows $f : W \to Z^{D^n}$ and $g : W \to Z^*$ such that $i^*(f) = h^*(g)$ (so $f \circ i = g \circ h$), we would like to show there exists a unique map $\ell : W \to Z^{S^n}$ such that $j^*(\ell) = f$ and $k^* \circ \ell = g$. For each $w \in W$, we have a map $f(w) : D^n \to Z$ and $g(w) : * \to Z$ such that $f(w) \circ i = g(w) \circ h$ which make the following diagram commute

$$\begin{tikzcd}
S^{n-1} \arrow[r, "i"] \arrow[d, "h"'] & D^n \arrow[d, "j"] \arrow[ddr, bend left, "f(w)"] & \\
* \arrow[r, "k"] \arrow[drr, bend right, "g(w)"'] & S^n \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "\ell(w)"'] & \\
 & & Z
\end{tikzcd}$$

Hence by the universal property of the pushout, there is a unique map $\ell(w) : S^n \to Z$ such that $\ell(w) \circ j = f(w)$ and $\ell(w) \circ k = g(w)$. In this way, we have defined a function $\ell : W \to Z^{S^n}$, and furthermore it clearly makes the following diagram commute

$$\begin{tikzcd}
W \arrow[dr, "\ell"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& Z^{S^n} \arrow[r, "j^*"] \arrow[d, "k^*"'] & Z^{D^n} \arrow[d, "i^*"] \\
& Z^* \arrow[r, "h^*"] & Z^{S^{n-1}}
\end{tikzcd} \tag{8}$$

Furthermore, it is not difficult to see that ℓ is the *unique* function $W \to Z^{S^n}$ which makes this diagram commute, as necessarily given $w \in W$, we know $\ell(w) : S^n \to Z$ is a continuous map satisfying $\ell(w) \circ j = f(w)$ and $\ell(w) \circ k = g(w)$, but since (6) is a pushout diagram, $\ell(w)$ is the *unique* such map, so ℓ must be defined as we defined it. It remains to show that ℓ is continuous. By definition of the compact-open topology, it suffices to show that for all compact $K \subseteq S^n$ and open $U \subseteq Z$ that

$$\ell^{-1}(S(K,U)) = \{w \in W : \ell(w)(K) \subseteq U\}$$

is open in W. Note it suffices to show that $\ell(w)(K) = f(w)(j^{-1}(K))$ and that $j^{-1}(K)$ is compact, as if this were true then we would have

$$\ell^{-1}(S(K,U)) = \{w \in W : \ell(w)(K) \subseteq U\} = \{w \in W : f(w)(j^{-1}(K)) \subseteq U\} = f^{-1}(S(j^{-1}(K), U)),$$

and since f is continuous, if $j^{-1}(K)$ is compact then $S(j^{-1}(K), U)$ is open in Z^{D^n}, so we would have that $\ell^{-1}(S(K,U)) = f^{-1}(S(j^{-1}(K), U))$ is open, as desired.

Now, to see $\ell(w)(K) = f(w)(j^{-1}(K))$, note that given $z \in \ell(w)(K)$, there exists $x \in K \subseteq S^n$ such that $\ell(w)(x) = z$, then since j is surjective we may pick $d \in D^n$ such that $j(d) = x \in K$. Furthermore, $z = \ell(w)(x) = \ell(w)(j(d)) = f(w)(d)$. Thus it follows $z \in f(w)(j^{-1}(K))$. Conversely, suppose we are given $z \in f(w)(j^{-1}(K))$, so there exists $d \in D^n$ such that $z = f(w)(d) = \ell(w)(j(d))$, and $j(w) \in K$. Thus it follows $z \in \ell(w)(K)$, as desired.

Finally, to see $j^{-1}(K)$ is compact, note that j is a continuous map between compact Hausdorff spaces, so that $j^{-1}(K)$ is compact in D^n by Lemma 2.20. Thus, we have shown that diagram (7) is a pullback square.

Now, by Corollary 2.17 we know that the map $i^* : Z^{D^n} \to Z^{S^{n-1}}$ in diagram (7) belongs to $J_\perp$. Thus since $J_\perp$ is characterized by a right lifting property, it is straightforward to see it is closed under taking pullbacks, so that the left map in the diagram $k^* : Z^{S^n} \to Z^*$ also belongs to $J_\perp$. Under the obvious isomorphism $Z^* \cong Z$, this is precisely the evaluation map $Z^{S^n} \to Z$ sending $\alpha \mapsto \alpha(*)$. □

Lemma 2.22 (Hovey 2.4.17). *Suppose $p : X \to Y$ is a weak equivalence. Then $p^{S^n} : X^{S^n} \to Y^{S^n}$ is a weak equivalence for all $n \geq -1$, where $S^{-1} = \emptyset$.*

Proof. We give a proof by induction on n. In the case $n = -1$, since $S^{-1} = \emptyset$, $X^{S^{-1}} = Y^{S^{-1}} = *$ (because $\emptyset$ is initial in **Top**), so that the map $p^{S^{-1}} : * \to *$ is a homeomorphism (because $*$ is terminal) and in particular a weak-equivalence.

Now, supposing we have shown $p^{S^{n-1}}$ is a weak equivalence for some $n \geq 0$, we wish to show p^{S^n} is likewise. Let $i : S^{n-1} \to D^n$ and $j : D^n \to S^n$ be as in diagram (6). Pick a point $\alpha \in X^{S^n}$, let F_X denote the fiber of $i_X : X^{D^n} \to X^{S^{n-1}}$ containing $\alpha \circ j$ and let F_Y denote the corresponding fiber of $i_Y : Y^{D^n} \to Y^{S^{n-1}}$ containing $p \circ \alpha \circ j$. Let ι_X and ι_Y denote the inclusions $F_X \hookrightarrow X^{D^n}$ and $F_Y \hookrightarrow Y^{D^n}$, respectively. Then we first claim that $F_X \to F_Y$ is a weak equivalence. First, we claim $\pi_0(p_*) : \pi_0(F_X) \to \pi_0(F_Y)$ is a bijection. Since $p : X \to Y$ is a weak equivalence, it suffices to show that $\pi_0(F_X) \cong \pi_n(X, \alpha(*))$ and $\pi_0(F_Y) \cong \pi_n(Y, p(\alpha(*)))$, and that the following diagram commutes under these bijections

$$\begin{tikzcd}
\pi_0(F_X) \arrow[r] \arrow[d, "\cong"] & \pi_0(F_Y) \arrow[d, "\cong"'] \\
\pi_n(X,\alpha(*)) \arrow[r, "\cong"] & \pi_n(Y,p(\alpha(*)))
\end{tikzcd} \tag{9}$$

An element of $F_X = i_X^{-1}(i_X(\alpha\circ j))$ is a map $\beta : D^n \to X$ satisfying $\beta\circ i = \alpha\circ j\circ i$. Unravelling definitions, $\alpha\circ j\circ i : S^{n-1}\to X$ is simply the constant map on $\alpha(*)$, so that if $\beta : D^n \to X$ belongs to F_X, it must send every element in S^{n-1} to $\alpha(*)$. Thus β fits into the pushout square (6), so there exists a unique map $\widetilde{\beta} : S^n \to X$ such that the composition $D^n \xrightarrow{j} S^n \xrightarrow{\widetilde{\beta}} X$ is equal to β, and $\widetilde{\beta}(*) = \alpha(*)$. Conversely, an element of X^{S^n} sending $*$ to $\alpha(*)$ gives rise to an element of F_X by precomposition with j. Thus we may identify F_X (as a set) with the subset $[S^n, X]_* \subseteq X^{S^n}$ containing those maps $\beta : S^n \to X$ such that $\beta(*) = \alpha(*)$. We further claim the assignment $h_X : F_X \to [S^n, X]_*$ is a homeomorphism, where $[S^n, X]_*$ is endowed with the subspace topology inherited from X^{S^n}. Given $K \subseteq S^n$ compact and $U \subseteq X$ open, we want to show that $h_X^{-1}(S(K,U))$ is open in F_X. Indeed,

$$h_X^{-1}(S(K,U)\cap[S^n,X]_*) = \{\beta\in F_X : \widetilde{\beta}(K)\subseteq U\} \overset{(*)}{=} \{\beta\in F_X : \beta(j^{-1}(K))\subseteq U\} = S(j^{-1}(K),U)\cap F_X,$$

where $(*)$ follows by the fact that $j : D^n \twoheadrightarrow S^n$ is surjective and $\widetilde{\beta}\circ j = \beta$. Note $j^{-1}(K)$ is compact in D^n by Lemma 2.20, so that h_X is continuous as desired. Conversely to see h_X is open note a similar argument yields

$$h_X(S(K,U)\cap F_X) = S(j(K),U)\cap[S^n,X]_*$$

for all $K \subseteq D^n$ compact and $U \subseteq X$ open (continuous maps preserve open maps), so that h_X is a continuous open bijection, therefore a homeomorphism, as desired. Thus the assignment $\beta\mapsto\widetilde{\beta}$ yields a bijection between $\pi_0(F_X) \to \pi_0([S^n,X]_*) = \pi_n(X,\alpha(*))$, i.e., the path component of some $\beta \in F_X \subseteq X^{D^n}$ is sent to the equivalence class of the map $\widetilde{\beta} : S^n \to X$ in $\pi_n(X,\alpha(*))$. An entirely analagous argument yields a bijection $\pi_0(F_Y) \to \pi_n(Y,p(\alpha(*)))$ sending the path component of a map $\gamma : D^n \to Y$ in F_Y to the equivalence class of the unique continuous map $\widetilde{\gamma} : (S^n,*) \to (Y,p(\alpha(*)))$ such that $\gamma = \widetilde{\gamma}\circ j$. Finally, to see diagram (9) commutes, unravelling the maps we have that the top composition sends $[\beta]\in\pi_0(F_X)$ to $[\widetilde{p\circ\beta}]$, while the bottom composition sends it to $[p\circ\widetilde{\beta}]$. By definition, $\widetilde{p\circ\beta}$ is the unique continuous dashed line $S^n \to Y$ such that the following diagram commutes

$$\begin{tikzcd}
S^{n-1} \arrow[r, "i"] \arrow[d, "h"'] & D^n \arrow[d, "j"] \arrow[ddr, bend left, "p\circ\beta"] & \\
* \arrow[r, "k"] \arrow[drr, bend right, "p(\alpha(*))"'] & S^n \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed] & \\
 & & Y
\end{tikzcd}$$

Then it follows by commutativity of the following diagram that $p \circ \widetilde{\beta} = \widetilde{p \circ \beta}$

$$\begin{tikzcd}
S^{n-1} \arrow[r, "i"] \arrow[d, "h"'] & D^n \arrow[d, "j"] \arrow[ddr, bend left, "\beta"] \arrow[dddrr, bend left, "p\circ\beta"] & & \\
* \arrow[r, "k"] \arrow[drr, bend right, "\alpha(*)"] \arrow[ddrrr, bend right, "p(\alpha(*))"'] & S^n \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "\widetilde{\beta}"] & & \\
& & X \arrow[dr, dashed, "p"] & \\
& & & Y
\end{tikzcd}$$

Thus since diagram (9) commutes, we get that $\pi_0(p_*) : \pi_0(F_X) \to \pi_0(F_Y)$ is a bijection, as desired.

Now, we would like to show that $\pi_m(p_*, \alpha \circ j) : \pi_m(F_X, \alpha \circ j) \to \pi_m(F_Y, p \circ \alpha \circ j)$ is a bijection for all $m > 0$. Consider the square

$$\begin{tikzcd}
X^{D^n} \arrow[r, "p^{D^n}"] \arrow[d, "i_X"'] & Y^{D^n} \arrow[d, "i_Y"] \\
X^{S^{n-1}} \arrow[r, "p^{S^{n-1}}"] & Y^{S^{n-1}}
\end{tikzcd}$$

It commutes, as given $f \in X^{D^n}$, $i_Y(p^{D^n}(f)) = i_Y(p \circ f) = p \circ f \circ i = p \circ i_X(f) = p^{S^{n-1}}(i_X(f))$. Furthermore, i_X and i_Y belong to $J_\perp$ by Corollary 2.17. Thus by Lemma 2.19, for all $m > 0$ the following diagram commutes and both rows are exact

$$\begin{tikzcd}
\pi_{m+1}(X^{D^n}, \alpha j) \arrow[r, "\pi_{m+1}(i_X, \alpha j)"] \arrow[d, "\pi_{m+1}(p^{D^n}, \alpha j)"] & \pi_{m+1}(X^{S^{n-1}}, \alpha j i) \arrow[r, "d_m"] \arrow[d, "\pi_{m+1}(p^{S^{n-1}}, \alpha j i)"] & \pi_m(F_X, \alpha j) \arrow[r, "\pi_m(\iota_X, \alpha_j)"] \arrow[d, "\pi_m(p^{D^n}, \alpha j)"] & \pi_m(X^{D^n}, \alpha j) \arrow[r, "\pi_m(i_X, \alpha j)"] \arrow[d, "\pi_m(p^{D^n}, \alpha j)"] & \pi_m(X^{S^{n-1}}, \alpha j i) \arrow[d, "\pi_m(p^{S^{n-1}}, \alpha j i)"] \\
\pi_{m+1}(Y^{D^n}, p\alpha j) \arrow[r, "\pi_{m+1}(i_Y, \alpha j)"'] & \pi_{m+1}(Y^{S^{n-1}}, p\alpha j i) \arrow[r, "d_m"'] & \pi_m(F_Y, p\alpha j) \arrow[r, "\pi_m(\iota_Y, \alpha_j)"'] & \pi_m(Y^{D^n}, p\alpha j) \arrow[r, "\pi_m(i_Y, \alpha j)"'] & \pi_m(Y^{S^{n-1}}, p\alpha j i)
\end{tikzcd}$$

Now per our induction hypothesis, the second and fifth vertical arrows are isomorphisms. The first and fourth vertical arrows are isomorphisms by Lemma 2.18. Then it follows by the five-lemma that the middle arrow is an isomorphism (in the case $m = 1$, this argument still works, a simple diagram chase yields that the five-lemma does hold in the more general setting when the last three elements of each row are non-abelian groups). Thus the restriction $p^{D^n} : F_X \to F_Y$ is indeed a weak equivalence, as desired.

In what follows, fix a basepoint $* \in S^n$, and given a space Z let $k_Z : Z^{S^n} \to Z$ be the evaluation map sending $\alpha \mapsto \alpha(*)$. let $\widetilde{F_X}$ and $\widetilde{F_Y}$ be the fibers of $k_X : X^{S^n} \to X$ and $k_Y : Y^{S^n} \to Y$ containing α and $p \circ \alpha$, respectively. Let F_X and F_Y be defined as above (so $p^{D^n} : F_X \to F_Y$ is a weak equivalence, as we have just shown). Finally, let $j_Z : Z^{S^n} \to Z^{D^n}$ be the pullback map induced by $j : D^n \to S^n$. Consider the following square

(10)
$$\begin{tikzcd}
\widetilde{F_X} \arrow[r, "j_X"] \arrow[d, "p^{S^n}"'] & F_X \arrow[d, "p^{D^n}"] \\
\widetilde{F_Y} \arrow[r, "j_Y"] & F_Y
\end{tikzcd}$$

First of all, it clearly commutes, as given $\beta \in \widetilde{F_X}$, so $\beta : S^n \to X$ is continuous and $\beta(*) = \alpha(*)$, we have $p^{D^n}(j_X(\beta)) = p^{D^n}(\beta \circ j) = p \circ \beta \circ j = j_Y(p \circ \beta) = j_Y(p^{S^n}(\beta))$. Now we claim the restriction $j_X|_{\widetilde{F_X}} : \widetilde{F_X} \to F_X$ is a homeomorphism. First, note explicitly that

$$F_X = i_X^{-1}(i_X(\alpha \circ j)) = \{\gamma \in X^{D^n} : i_X(\gamma) = i_X(\alpha \circ j)\} = \{\gamma \in X^{D^n} : \gamma \circ i = \alpha \circ j \circ i\},$$

and

$$\widetilde{F_X} = k_X^{-1}(k_X(\alpha)) = \{\beta \in X^{S^n} : k_X(\beta) = k_X(\alpha)\} = \{\beta \in X^{S^n} : \beta(*) = \alpha(*)\}.$$

Now we show $j_X : \widetilde{F_X} \to F_X$ is a homeomorphism by considering the cases $n = 0$ and $n > 0$ separately.

In the case $n = 0$, given $\gamma \in X^{D^0}$, $\gamma \circ i$ and $\alpha \circ j \circ i$ are maps $S^{-1} \to X$, and $S^{-1} = \emptyset$ is initial in **Top**, so $\gamma \circ i = \alpha \circ j \circ i$ for all $\gamma \in X^{D^0}$. Thus $F_X = X^{D^0} = X^*$. On the other hand, an element of $\widetilde{F_X}$ is a function $\beta : S^0 = \{*, 0\} \to X$ such that $\beta(*) = \alpha(*)$, and the map $j_X : \widetilde{F_X} \to F_X$ sends $\beta \mapsto \beta \circ j$. Here j is simply the map $D^0 \to \{*, 0\}$ including the unique point $0 \in D^0$ to 0. Thus under the isomorphism $X^* \cong X$, $j_X : \widetilde{F_X} \to F_X = X^* \cong X$ simply sends $\beta \mapsto \beta(0)$. Now, clearly this map is a bijection, as an element of $\widetilde{F_X}$ is a function $\{*, 0\} \to X$ which must send $*$ to $\alpha(*)$ and there are no restrictions on where 0 is sent. To see this map is open, let $K \subseteq S^0$ compact (so K is any subset of S^0 as S^0 is a finite space) and let $U \subseteq X$ be open. Then we would like to show $j_X(S(K, U) \cap \widetilde{F_X})$ is open in X. Note

$$j_X(S(K,U) \cap \widetilde{F_X}) = \{\beta(0) : \beta \in X^{\{*,0\}}, \beta(*) = \alpha(*), \beta(K) \subseteq U\} = \begin{cases} U & * \in K, \alpha(*) \in U \\ \emptyset & * \in K, \alpha(*) \notin U \\ U & * \notin K. \end{cases}$$

Hence $j_X(S(K, U) \cap \widetilde{F_X})$ is indeed open. Thus in the case $n = 0$, $j_X : \widetilde{F_X} \to F_X$ is an open bijection, and it is continuous bijection, so it is a homeomorphism.

Now, suppose that $n > 0$, we would like to show $j_X : \widetilde{F_X} \to F_X$ is a continuous open bijection. It is continuous by definition. Since j is an epimorphism when $n > 0$, j_X is clearly injective, as given $\beta, \beta' \in \widetilde{F_X}$, if $j_X(\beta) = j_X(\beta')$, then $\beta \circ j = \beta' \circ j \implies \beta = \beta'$, as desired. To see it is surjective, first note that $j \circ i : S^{n-1} \to D^n \to S^n$ sends everything to the basepoint $*$, by commutativity of (6). Thus elements of F_X are maps $\gamma : D^n \to X$ such that $\gamma \circ i : S^{n-1} \hookrightarrow D^n \to X$ sends everything to $\alpha(*)$, i.e., γ restricts to the constant map on $\alpha(*)$ on the boundary $\partial D^n = S^{n-1}$. Thus γ fits into the following diagram

$$\begin{tikzcd}
S^{n-1} \arrow[r, "i"] \arrow[d, "h"'] & D^n \arrow[d, "j"] \arrow[ddr, bend left, "\gamma"] & \\
* \arrow[r, "k"] \arrow[ur, phantom, "\ulcorner", very near end] \arrow[drr, bend right, "\alpha(*)"'] & S^n \arrow[dr, dashed, "\widetilde{\gamma}"'] & \\
& & X
\end{tikzcd}$$

so there is a (unique) dashed arrow $\widetilde{\gamma} : S^n \to X$ such that $j_X(\widetilde{\gamma}) = \widetilde{\gamma} \circ j = \gamma$, and $\widetilde{\gamma} \in \widetilde{F_X}$ as $(\widetilde{\gamma}(*)) = \alpha(*)$. Thus j_X is surjective. It remains to show that j_X is open. Let $K \subseteq S^n$ be compact and $U \subseteq X$ open. Then we would like to show that $j_X(S(K, U) \cap \widetilde{F_X})$ is open in F_X. It suffices to show that $j_X(S(K, U) \cap \widetilde{F_X}) = F_X \cap S(j^{-1}(K), U)$, as $j^{-1}(K)$ is compact (Lemma 2.20). First of all, let $\beta \in S(K, U) \cap \widetilde{F_X}$. We would like to show $j_X(\beta) = \beta \circ j \in S(j^{-1}(K), U)$. Indeed, $\beta \circ j(j^{-1}(K)) = \beta(j(j^{-1}(K))) = \beta(K) \subseteq U$, where the second equality follows by the fact that j is surjective (since $n > 0$). Conversely, suppose we are given $\gamma \in S(K, U) \cap \widetilde{F_X}$. Then by what we have shown above, $\gamma = \widetilde{\gamma} \circ j$ for a unique $\widetilde{\gamma} \in X^{S^n}$ with $\widetilde{\gamma}(*) = \alpha(*)$, and $\gamma(j^{-1}(K)) \subseteq U$, so that $\widetilde{\gamma}(j(j^{-1}(K))) = \widetilde{\gamma}(K) \subseteq U$ (where again equality follows since j is surjective when $n > 0$). Thus $\widetilde{\gamma} \in S(K, U) \cap \widetilde{F_X}$ and satisfies $j_X(\widetilde{\gamma}) = \widetilde{\gamma} \circ j = \gamma$, so $\gamma \in j_X(S(K, U) \cap \widetilde{F_X})$. Hence, j_X is a continuous open bijection, thus a homeomorphism, as desired.

By an entirely analagous argument, $j_Y : \widetilde{F_Y} \to F_Y$ is a homeomorphism. Thus looking at diagram (10), since the top and bottom arrows are isomorphisms and we have shown the right arrow is a weak equivalence, it follows by two applications of 2-of-3 (Lemma 2.12) that the left arrow $p^{S^n} : \widetilde{F_X} \to \widetilde{F_Y}$ is a weak equivalence as well.

Now we will finally show that $\pi_m(p^{S^n}, \alpha) : \pi_m(X^{S^n}, \alpha) \to \pi_m(Y^{S^n}, p \circ \alpha)$ is a bijection for all $m \geq 0$. Consider the square

$$\begin{tikzcd}
X^{S^n} \arrow[r, "p^{S^n}"] \arrow[d, "k_X"'] & Y^{S^n} \arrow[d, "k_Y"] \\
X \arrow[r, "p"] & Y
\end{tikzcd}$$

This diagram commutes, as given $\beta \in X^{S^n}$, $k_Y(p_*(\beta)) = k_Y(p \circ \beta) = p(\beta(*)) = p(k_X(\beta))$. Redefine ι_X and ι_Y to be the inclusions $\widetilde{F_X} \hookrightarrow X^{S^n}$ and $\widetilde{F_Y} \hookrightarrow Y^{S^n}$, respectively. Since k_X and k_Y are fibrations, by Lemma 2.19, for all $m \geq 0$ the following diagram commutes and the rows are exact

$$\begin{tikzcd}
\pi_{m+1}(X, \alpha(*)) \arrow[r, "d_m"] \arrow[d, "\pi_{m+1}(p, \alpha(*))"] & \pi_m(\widetilde{F_X}, \alpha) \arrow[r, "\pi_m(\iota_X, \alpha)"] \arrow[d, "\pi_m(p^{S^n}, \alpha)"] & \pi_m(X^{S^n}, \alpha) \arrow[r, "\pi_m(k_X, \alpha)"] \arrow[d, "\pi_m(p^{S^n}, \alpha)"] & \pi_m(X, \alpha(*)) \arrow[r, "d_{m-1}"] \arrow[d, "\pi_m(p, \alpha(*))"] & \pi_{m-1}(\widetilde{F_X}, \alpha) \arrow[d, "\pi_{m-1}(p^{S^n}, \alpha)"] \\
\pi_{m+1}(Y, p(\alpha(*))) \arrow[r, "d_m"'] & \pi_m(\widetilde{F_Y}, p\alpha) \arrow[r, "\pi_m(\iota_Y, p\alpha)"'] & \pi_m(Y^{S^n}, p\alpha) \arrow[r, "\pi_m(k_Y, p\alpha)"'] & \pi_m(Y, p(\alpha(*))) \arrow[r, "d_{m-1}"'] & \pi_{m-1}(\widetilde{F_Y}, p\alpha)
\end{tikzcd}$$

(in the case $m = 0$, the final entry of each row becomes 0). We know the second and fifth vertical arrows are isomorphisms by what we have shown above. Since p is a weak equivalence a priori, we also have that the first and fourth vertical arrows are isomorphisms. Thus by the five-lemma we get that the middle arrow is an isomorphism when $m > 0$ (again, a simple diagram chase yields that the five-lemma still works here in the case $m = 1$, as it holds in the more general setting where the first and second entries are abelian groups, the third and fourth entries are non-abelian groups, and the last entry of each row is a set), as desired.

Finally, we claim that $\pi_0(p^{S^n}, \alpha) : \pi_0(X^{S^n}, \alpha) \to \pi_0(Y^{S^n}, p \circ \alpha)$ is a bijection. to see surjectivity, we first claim that $\pi_0(X^{S^n}) \cong [S^n, X]$ where $[A, Z]$ means the set of (free) homotopy classes of maps from A to Z. Indeed, connected components of X^{S^n} are precisely the homotopy classes of maps $S^n \to X$. In order to see this, first suppose $\beta, \gamma \in X^{S^n}$ are in the same connected component, so there exists a continuous map $f : I \to X^{S^n}$ with $f(0) = \beta$ and $f(1) = \gamma$. Since S^n is LCH (in particular it is compact and Hausdorff), we get an induced map $F : I \times S^n \to X$ (Proposition 2.1). Unravelling how this map is defined, we have $F(t, s) = f(t)(s)$ for all $t \in I$ and $s \in S^n$, so that in particular $F(0, s) = \beta(s)$ and $F(1, s) = \gamma(s)$, so F defines a homotopy between β and γ, as desired. Conversely, given two maps $\beta, \gamma : S^n \to X$ and a homotopy $H : I \times S^n \to X$ with $H(0, s) = \alpha(s)$ and $H(1, s) = \beta(s)$ for all $s \in S^n$, again since S^n is LCH, H induces a map $h : I \to X^{S^n}$, and $h(0) = H(0, -) = \beta$ and $h(1) = H(1, -) = \gamma$, so that β and γ belong to the same path-component of X^{S^n}, as desired. Similarly, $\pi_0(Y^{S^n}) \cong [S^n, Y]$. Thus in order to show $\pi_0(p^{S^n})$ is surjective it suffices to show that the map $p_* : [S^n, X] \to [S^n, Y]$ is surjective. Indeed, let $[f] \in [S^n, Y]$. Since $\pi_0(p)$ is bijective, there exists $x \in X$ and a path $\gamma : I \to Y$ with $\gamma(0) = p(x)$ and $\gamma(1) = f(*)$. Then conjugation by γ yields an isomorphism $h_\gamma : \pi_n(Y, f(*)) \cong \pi_n(Y, p(x))$ which preserves free homotopy equivalence, and since p is a weak equivalence there exists $[g] \in \pi_n(Y, p(x))$ and a homotopy between $h_\gamma([f])$ and $[p \circ g]$, so it follows $[p \circ g]$ is homotopic to $[f]$ as desired (see a similar discussion in the proof of Lemma 2.12).

Finally, to see injectivity of $\pi_0(p^{S^n})$, suppose we are given two points $\beta, \gamma \in X^{S^n}$ which are sent to the same path component of Y^{S^n}. Note that our choice of $\alpha \in X^{S^n}$ all the way above was arbitrary, and we could have chosen α to be any element, so WLOG let's assume $\alpha = \gamma$. Then chasing $[\alpha]$ and $[\beta]$ around the third square in the diagram above yields that p sends $\alpha(*)$

and $\beta(*)$ to the same path component in Y, and $\pi_0(p)$ is a bijection, so it follows that $\alpha(*)$ and $\beta(*)$ belong to the same path component of X. $[\alpha(*)] = [\beta(*)]$ is the distinguished point of $\pi_0(X, \alpha(*))$, so by exactness it follows that there exists $\widetilde{\beta} \in \widetilde{F_X}$ such that $[\beta] = [\iota_X(\widetilde{\beta})]$. Then chasing $\widetilde{\beta}$ around the second square yields that $[p(\widetilde{\beta})]$ is in the "kernel" of $\pi_0(\iota_Y, p\alpha)$, so that there exists $\delta \in \pi_1(Y, p(\alpha(*)))$ such that $d_0([\delta]) = [p(\widetilde{\beta})]$. Then since $\pi_1(p, \alpha(*))$ is surjective, there further exists $\delta' \in \pi_1(X, \alpha(*))$ such that $[p(\delta')] = [\delta]$. Finally, chasing δ' around the first arrow yields that

$$\pi_0(p^{S^n}, \alpha)(d_0([\delta'])) = d_0([p(\delta')]) = d_0([\delta]) = [p(\widetilde{\beta})] = \pi_0(p^{S^n}, \alpha)([\widetilde{\beta}]),$$

and $\pi_0(p^{S^n}, \alpha) : \pi_0(\widetilde{F_X}, \alpha) \to \pi_0(\widetilde{F_Y}, p\alpha)$ is a bijective, so that $d_0([\delta']) = [\widetilde{\beta}]$. Thus it follows by exactness that $[\beta] = \pi_0(\iota_X, \alpha)([\widetilde{\beta}]) = \pi_0(\iota_X, \alpha)(d_0([\delta'])) = [\alpha]$, as desired. ☐

Proposition 2.23 (Hovey 2.4.18). *Suppose we have a pullback square*

$$\begin{tikzcd}
W \arrow[r, "f"] \arrow[d, "q"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "p"] \\
Z \arrow[r, "g"] & Y
\end{tikzcd}$$

in **Top***, where $p \in J_\perp$ and g is a weak equivalence. Then f is a weak equivalence.*

Proof. Let $w \in W$, and define $F := q^{-1}(q(w))$, $F' := p^{-1}(p(f(w)))$. We claim f restricts to a homeomorphism $F \to F'$. First of all clearly given $a \in F = q^{-1}(q(w))$ (so $q(a) = q(w)$), note $p(f(a)) = g(q(a)) = g(q(w)) = p(f(w))$ so that $f(a) \in p^{-1}(p(f(w)))$, as desired. To see it is injective, suppose we are given $a, b \in F$ such that $f(a) = f(b)$. Then consider the maps $h : * \to X$ and $k : * \to Z$ sending $* \mapsto f(a)$ and $* \mapsto q(w)$, respectively. Clearly

$$p(h(*)) = p(f(a)) = g(q(a)) = g(q(w)) = g(k(*)).$$

Then by the universal property of the pullback, there exists a **unique** map $\ell : * \to W$ such that $f \circ \ell = h$ and $q \circ \ell = k$. It is straightforward to see that $\ell : * \mapsto a$ is a solution as $q(\ell(*)) = q(a) = q(w) = k(*)$ and $f(\ell(*)) = f(a) = h(*)$. Yet $\ell : * \mapsto b$ is also a solution, as $q(\ell(*)) = q(b) = q(w) = k$ and $f(\ell(*)) = f(b) = f(a) = h(*)$. Thus we must have had $a = b$ in the first place. To see it is surjective, suppose we are given $x \in F' = p^{-1}(p(f(w)))$. Consider the maps $h : * \to X$ and $k : * \to Z$ sending $* \mapsto x$ and $* \mapsto q(w)$, respectively. Clearly

$$p(h(*)) = p(x) = p(f(w)) = g(q(w)) = g(k(*)),$$

so that by the universal property of the pullback there exists a map $\ell : * \to W$ such that $f \circ \ell = h$ and $q \circ \ell = k$. Then $\ell(*) \in F = q^{-1}(q(w))$, as $q(\ell(*)) = k(*) = q(w)$, and $f(\ell(*)) = h(*) = x$. Thus we have found an element $\ell(*) \in F$ such that $f(\ell(*)) = x$, so $f : F \to F'$ is surjective, as desired.

Finally, it remains to show that f is open. By how limits are defined in **Top** (see the discussion at the beginning of this section), we know that W is in bijection with the space $\{(z, x) \in Z \times X : g(z) = p(x)\}$. In particular, there exists $z_0 \in Z$ and $x_0 \in X$ such that w corresponds to the point $(z_0, x_0) \in Z \times X$ under this bijection (so $g(z_0) = p(x_0)$). Furthermore, under this bijection q and f are simply the restriction of the projection maps $Z \times X \to Z$ and $Z \times X \to X$, respectively, and the topology on W has subbasis given by sets of the form $q^{-1}(U)$ and $f^{-1}(V)$ for $U \subseteq Z$ and $V \subseteq X$ open. In particular, F has subbasis given by sets of the form $q^{-1}(U) \cap F$ and $f^{-1}(V) \cap F$ for $U \subseteq Z$ and $V \subseteq X$ open. Now, in order to show $f|_F : F \to F'$ is open, it suffices to show it sends elements of the subbasis to open sets in F'. First note that $F = q^{-1}(q(w)) = q^{-1}(q(z_0, x_0)) = q^{-1}(z_0)$. Then given $U \subseteq Z$ open we have

$$f(q^{-1}(U) \cap F) = f(q^{-1}(U) \cap q^{-1}(z_0)) = f(q^{-1}(U \cap \{z_0\})) = \begin{cases} f(q^{-1}(z_0)) = f(F) \overset{(*)}{=} F' & z_0 \in U \\ \emptyset & z_0 \notin U, \end{cases}$$

where $(*)$ follows by the fact that $f(F) = F'$, as we showed above. Either way, $f(q^{-1}(U) \cap F)$ is open, as desired. Conversely, given an open set $V \subseteq X$, we have

$$f(f^{-1}(V) \cap F) \overset{(*)}{=} f(f^{-1}(V) \cap f^{-1}(F')) = f(f^{-1}(V \cap F')) \overset{(*)}{=} V \cap F,$$

where both ocurrences of $(*)$ follow by the fact that $f|_F : F \to F'$ is a bijection, as we showed above. By definition of the subspace topology, $V \cap F'$ is open in F'. Hence we have shown f sends elements of the subbasis of F to open sets in F', so it follows that $f|_F : F \to F'$ is properly open as desired.

Now, we apply Lemma 2.19 to get that for $n > 0$ the following diagram commutes and both rows are exact:

$$\begin{tikzcd}
\pi_{n+1}(Z, q(w)) \arrow[r, "d_n"] \arrow[d, "\pi_{n+1}(g, q(w))"] & \pi_n(F, w) \arrow[r, "\pi_n(\iota, w)"] \arrow[d, "\pi_n(f, w)"] & \pi_n(W, w) \arrow[r, "\pi_n(q, w)"] \arrow[d, "\pi_n(f, w)"] & \pi_n(Z, q(w)) \arrow[r, "d_{n-1}"] \arrow[d, "\pi_n(g, q(w))"] & \pi_{n-1}(F, w) \arrow[d, "\pi_{n-1}(f, w)"] \\
\pi_{n+1}(Y, p(f(w))) \arrow[r, "d_n"'] & \pi_n(F', f(w)) \arrow[r, "\pi_n(\iota', f(w))"'] & \pi_n(X, f(w)) \arrow[r, "\pi_n(p, f(w))"'] & \pi_n(Y, p(f(w)) \arrow[r, "d_{n-1}"'] & \pi_{n-1}(F', f(w))
\end{tikzcd}$$

Furthermore, by the five lemma, since g is a weak equivalence and $f|_F : F \to F'$ is a homeomorphism, it follows that the middle arrow is an isomorphism. It remains to show that $\pi_0(f) : \pi_0(W) \to \pi_0(X)$ is an isomorphism. The same argument using the trick of changing the basepoint in the last paragraph of the proof of Lemma 2.22 works to show that $\pi_0(f)$ is injective. To see it is surjective, suppose $x \in X$. Then since g is a weak equivalence, there is a point $z \in Z$ and a path $\gamma : I \to Y$ from $p(x)$ to $g(z)$. In other words, the following diagram commutes

$$\begin{tikzcd}
D^0 \cong * \arrow[r, "x"] \arrow[d, hook] & X \arrow[d, "p"'] \\
D^0 \times I \cong I \arrow[r, "\gamma"] & Y
\end{tikzcd}$$

Since the left arrow belongs to J and $p \in J_\perp$, there is a lift $\ell : I \to X$ such that $\ell(0) = x$ and $p(\ell(t)) = \gamma(t)$ for all $t \in I$. In particular, $p(\ell(1)) = \gamma(1) = g(z)$, so that under the identification $W \cong Z \times_Y X$ given above, there is a point $(z, \ell(1)) = w \in W$, and ℓ is a path between $f(w) = \ell(1)$ and x, so that f does indeed hit the path component of x. Hence we have shown $\pi_0(f)$ is a bijection, as desired. □

Proposition 2.24 (Hovey 2.4.12)**.** $\mathcal{W} \cap J_\perp \subseteq I'_\perp$

Proof. Let $p : X \to Y$ belong to $\mathcal{W} \cap J_\perp$. By Lemma 2.15, in order to show $p \in I'_\perp$ it suffices to show the map $Q(i,p)$ belongs to $\mathcal{W} \cap J_\perp$ for all boundary inclusions $i : S^{n-1} \hookrightarrow D^n$ for $n \geq 0$. Given such an i, consider the pullback diagram defining $P(i,p)$ and $Q(i,p)$

$$\begin{tikzcd}
X^{D^n} \arrow[dr, "Q(i,p)"] \arrow[drr, bend left, "p^{D^n}"] \arrow[ddr, bend right, "X^i"'] & & \\
& P(i,p) \arrow[r, "f"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y^{D^n} \arrow[d, "Y^i"] \\
& X^{S^{n-1}} \arrow[r, "p^{S^{n-1}}"] & Y^{S^{n-1}}
\end{tikzcd}$$

By Corollary 2.17, the right-hand vertical map (Y^i) belongs to $J_\perp$. By Lemma 2.22, the bottom horizontal map $(p^{S^{n-1}})$ belongs to $\mathcal{W}$. By Proposition 2.23 the top horizontal map (f) also belongs to $\mathcal{W}$. Finally using Lemma 2.18, we get that p^{D^n} also belongs to $\mathcal{W}$, so that by 2-of-3 (Lemma 2.12), we get that $Q(i,p)$ belongs to $\mathcal{W}$. Finally, $Q(i,p)$ belongs to $J_\perp$ by Lemma 2.16. □